BERTRAND CURVES AND HELICES

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Introduction. Every student of classical differential geometry meets early in his course the subject of Bertrand curves, discovered in 1850 by J. Bertrand.[1] A Bertrand curve is a curve such that its principal normals are the principal normals of a second curve. It is proved in most texts on the subject that the characteristic property of such a curve is the existence of a linear relation between the curvature and the torsion; the discussion appears as an application of the Frenet-Serret formulas.[2] The paper here presented originated in an attempt to specify exactly what Bertrand curves exist having a given linear relation between curvature and torsion, and how such curves may be found. It is certainly well known that a curve is determined by its curvature and torsion uniquely except as to its position in space, more precisely that a curve whose curvature and torsion are given functions of its arc is found by the integration of a Riccati equation.[3] It is, however, impossible to integrate the equation for given curvature and torsion except in some simple cases, usually such that the required curve is a helix. The general theorem of Lie is of interest and value, but it does not give very definite information about the Bertrand curves. J. A. Serret proved in 1850 that curves of given constant curvature and curves of given constant torsion can be found by quadratures. These two kinds of curves and helices are all in a sense Bertrand curves arising in exceptional cases of the linear relation between curvature and torsion, as will appear below. L. Bianchi proved[4] that Bertrand curves for a given linear relation can be found by quadratures, so that in these respects this paper has nothing to add. It is, however, here proved that we may find by quadratures a unique Bertrand curve for a given linear relation with an arbitrary spherical representation; that the coördinates of any point of this curve are expressed in a simple way in terms of the coördinates of certain curves of constant curvature and of constant torsion both of which have the same spherical representation as the Bertrand curve. The study and use of the spherical representation suggests the determination of the spherical representation of the helices of a sphere. This discussion is carried out in the second section of this paper and the spherical helices found from their spherical representation. It is shown that the projection of

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[1] For this and other historical references see Encyclopädie der Mathematischen Wissenschaften, vol. 3, D1,2, p. 82ff.

[2] E.g., Picard, *Traité d'Analyse*, 2d ed., vol. 1, p. 394.

[3] Sophus Lie, 1882.

[4] *Geometria Differenziale*, p. 31. See also Darboux, *Théorie Générale des Surfaces*, vol. 1, pp. 42–45.