FUNCTIONS OF SEVERAL VARIABLES AND ABSOLUTE CONTINUITY, I

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Introduction. We are concerned in this paper with certain properties of real- and complex-valued functions of n real variables which are "potential functions of their generalized derivatives"—a concept introduced by G. C. Evans [2, 3][1] for functions of two variables and readily extended to the case of n variables, $n \geqq 2$. We shall call these functions merely functions of class $\mathfrak{P}$, sacrificing in the interests of brevity the descriptiveness of Evans' terminology.

While the results which are described here are not without intrinsic interest, their chief importance for us lies in the uses to which they are put in researches to be described in subsequent papers: the author of the present paper has found them necessary for the study of partial differential equations by means of the theory of transformations in Hilbert space and the author of the following paper, *Functions of several variables and absolute continuity*, II (C. B. Morrey, Jr.) has found them necessary for certain investigations in the calculus of variations. From the point of view of both of these branches of research, the importance of functions of class $\mathfrak{P}$ lies in the fact that they arise in the following way: Let $\{f_k(x_1, \cdots, x_n\}$ be a sequence of functions (real- or complex-valued) of class C′ on an open set G in n-dimensional space, and let the sequences $\{f_k\}$, $\{\partial f_k/\partial x_j\}$ $(j = 1, 2, \cdots, n)$ converge in the mean of order p $(p \geqq 1)$ on G. Then the limit (in the mean) f of the sequence $\{f_k\}$ is of class $\mathfrak{P}$ and the limits of the sequences of partial derivatives are the generalized derivatives of f in the sense of Evans. This result gives rise to several interesting questions. Most important, while the function f described above can evidently be arbitrarily defined on any set of measure zero, and thus cannot be expected to have partial derivatives in the ordinary sense, it is natural to seek to use this very freedom to redefine the function on a set of measure zero so as to obtain a function in some sense differentiable. It is shown below that the problem which arises in this connection has an entirely satisfactory solution; any function of class $\mathfrak{P}$ is equal almost everywhere to a function f_0 which has the following property: on almost all lines parallel to the x_j-axis and intersecting G, f_0 is absolutely continuous on every closed interval interior to G $(j = 1, \cdots, n)$. Functions of class $\mathfrak{P}$ which have this property are introduced below as functions of class $\mathfrak{P}'$.

We may also mention here that the convergence theorem stated above remains valid if the functions f_k are required merely to be of class $\mathfrak{P}$, and the sequences of derivatives are replaced by sequences of generalized derivatives. This im-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.