MONOTONE COVERINGS AND MONOTONE TRANSFORMATIONS

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1. **Introduction.** The purpose of this paper is to relate coverings with connected sets and transformations whose inverse sets are connected. We shall use the term *covering* in a restricted sense. All coverings shall be with *closed sets* and the number of sets in a covering shall be *finite* unless it is explicitly stated that the sets arise as *inverse sets of points* in connection with a continuous transformation. It is also supposed that the term *space* shall mean *compact metric space* and the term *transformation* shall imply *continuity* and *single-valuedness.*

Let $C: A = \sum A_\alpha$ be a covering of the space A. We may associate with C an abstract complex called the *nerve*[1] of C in such a fashion that the vertices of this complex correspond in a one-to-one way with the sets of the covering. If we let $N(C) = K$ denote this correspondence, then the vertices $N(A_{\alpha_0}), \cdots,$ $N(A_{\alpha_n})$ are spanned by an n-simplex if and only if the sets $A_{\alpha_0}, \cdots, A_{\alpha_n}$ have a non-vacuous product. We shall say that two coverings are *equivalent* if they are in one-to-one correspondence in such a way that the property of intersecting is preserved. The *dimension* of C is the largest integer n such that $n + 1$ different sets of C intersect. The words *chain, simple chain, simple closed chain,* and *acyclic* have their usual meanings and will be applied both to the covering and to its nerve. The covering C is said to be a *monotone covering* if the sets of C are *connected.* This terminology is by way of analogy with monotone transformation.

The next two definitions are generalizations of the notion of a fixed-point free transformation and are due to H. Hopf.[2] The covering $C: A = \sum A_\alpha$ is said to be *free* provided there exists a continuous transformation f of A into[3] itself such that $f(A_\alpha)A_\alpha = 0$ for each α. A transformation $T(A) = B$ is *free* if the covering of A with the sets $T^{-1}(y)$, $y \in B$, is free. This last is equivalent, as Hopf points out, to the condition that there exist a continuous transformation $f(A) \subset A$ such that $Tf(x) \neq T(x)$ for each $x \in A$. It is clear that if A does not admit a fixed-point free transformation, then it does not admit a free covering or a free transformation of any sort. Or, in other words, if every transformation of A into itself has a fixed-point, then A has no free covering and it is not

Received March 28, 1939; presented to the American Mathematical Society, December 27, 1938.

[1] Alexandroff-Hopf, *Topologie* I, Berlin, 1936, p. 152.

[2] H. Hopf, *Freie Überdeckungen und freie Abbildungen,* Fundamenta Mathematicae, vol. 28(1937), p. 31. All references to Hopf are to this paper. Both the methods and results of the present paper are intimately related to Hopf's work.

[3] If $T(A) \subset B$, then T maps A *into* B. We use *onto* if $T(A) = B$.