AN INVERSION FORMULA FOR THE LAPLACE INTEGRAL

By R. P. Boas, Jr. and D. V. Widder

Introduction. A function $f(s)$ which is represented by a Laplace-Stieltjes integral

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t), \tag{1}$$

being analytic in a half-plane $\sigma > \sigma_c$ $(s = \sigma + i\tau)$, is uniquely determined by its values in certain parts of that domain. We name four cases:

(a) the values of $f(s)$ and all its derivatives at a single point s_0, $\sigma_0 > \sigma_c$;

(b) the values of $f(s)$ and all its derivatives on the axis of reals in a neighborhood of infinity, $\tau = 0$, $\sigma > \sigma_1$;

(c) the values of $f(s)$ on a vertical line $\sigma = c$, $-\infty < \tau < \infty$;

(d) the values of $f(s)$ on the axis of reals $\tau = 0$, $\sigma_c < \sigma < \infty$.

In any of these cases it should be possible then to determine $\alpha(t)$ uniquely in terms of the stated values of $f(s)$ or its derivatives. The first case has been treated by use of Laguerre polynomials.[1] The second case is handled by the Post-Widder inversion formula[2]

$$\alpha(t) - \alpha(0+) = \lim_{k\to\infty} \int_{k/t}^\infty (-1)^{k+1} f^{(k+1)}(u) \frac{u^k}{k!}\, du.$$

Case (c) is the classical case:

$$\alpha(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} f(s) \frac{e^{st}}{s}\, ds.$$

Case (d) has been treated by Paley and Wiener[3] and by Doetsch.[4] It is the object of the present paper to provide a new inversion formula for this case.

Received December 15, 1939. The first-mentioned author is a National Research Fellow.

[1] D. V. Widder, *An application of Laguerre polynomials*, this Journal, vol. 1(1935), pp. 126–136.

A. G. Domínguez, *Sur les intégrales de Laplace*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, vol. 205(1937), pp. 1035–1038.

[2] E. L. Post, *Generalized differentiation*, Transactions of the American Mathematical Society, vol. 32(1930), pp. 723–793.

D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, Transactions of the American Mathematical Society, vol. 36(1934), pp. 107–200.

[3] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, New York, 1934, p. 43.

[4] G. Doetsch, *Bedingungen für die Darstellbarkeit einer Funktion als Laplace-Integral und eine Umkehrformel für die Laplace-Transformation*, Mathematische Zeitschrift, vol. 42(1937), pp. 263–286.