INVARIANTS OF CERTAIN STOCHASTIC TRANSFORMATIONS: THE MATHEMATICAL THEORY OF GAMBLING SYSTEMS

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Introduction. The "Regellosigkeit" principle of von Mises has been shown to correspond in the mathematical theory of probability to the fact that certain transformations of infinite dimensional Cartesian space into itself are measure preserving. It is the purpose of this paper to investigate the behavior of such transformations on more general spaces. The theorems at the basis of this work are stated in the first section and applied to obtain results concerning the existence and independence of "Kollektivs" in the second and third sections. In §§4, 5, and 6 certain invariants of the transformations considered are obtained. Previous results on these transformations are shown to be special cases of these invariance theorems.

1. **Preliminary definitions and theorems.** In this section we shall define the concepts and state the theorems which are the basis of all the work of the later sections.

Definition 1. A collection $\mathfrak{F}_1$ of sets in a space Ω_1 is a *field* if $E_1 \epsilon \mathfrak{F}_1$ and $E_2 \epsilon \mathfrak{F}_1$ implies $E_1 + E_2 \epsilon \mathfrak{F}_1$ and $E_1 - E_1E_2 \epsilon \mathfrak{F}_1$.[1]

Definition 2. A collection $\mathfrak{B}_1$ of sets in a space Ω_1 is a *Borel field* if $\mathfrak{B}_1$ is a field and if $E_j \epsilon \mathfrak{B}_1$ $(j = 1, 2, \cdots)$ implies $\sum_{j=1}^{\infty} E_j \epsilon \mathfrak{B}_1$.

Definition 3. A *probability measure* is an additive, non-negative set function $P_1(E)$ defined on a field $\mathfrak{F}_1$ in a space Ω_1, with $P_1(\Omega_1) = 1$, such that $P_1(\sum_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} P_1(E_j)$ whenever $\{E_j\}$ is a sequence of disjunct sets belonging to $\mathfrak{F}_1$ whose sum is also in $\mathfrak{F}_1$.

Definition 4. A space Ω_1 in which a probability measure P_1 has been defined on a Borel field $\mathfrak{B}_1$ is a *probability space.*

Definition 5. A *measurable set* in a probability space Ω_1 is a set E such that $E \epsilon \mathfrak{B}_1$.

Definition 6. Let Ω_1 be a probability space and Ω_1' a space on which there is given a Borel field $\mathfrak{B}_1'$ of measurable sets. Let $\phi(x)$ be a single-valued function whose domain is Ω_1 and whose range is in Ω_1'. $\phi(x)$ is a *measurable function* if the set E of points $x \epsilon \Omega_1$ for which $\phi(x)$ is in $E' \subseteq \Omega_1'$ is measurable whenever E' is.[2] If $\phi(x)$ is real valued, it is measurable if for every real number λ the

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[1] All the fields used in this paper will also satisfy the condition that if $E \epsilon \mathfrak{F}_1$, then $CE \epsilon \mathfrak{F}_1$, where CE is the complement in Ω_1 of the set E.

[2] The symbol $\{\phi(x) \epsilon E'\}$ will be used to denote E.