OSCILLATING FUNCTIONS

By R. P. Boas, Jr.

1. **Introduction.** We may say that a function $f(x)$ is monotonic at the point x_0 if there is a positive δ such that, whenever $x_0 - \delta < x_1 \leqq x_0 \leqq x_2 < x_0 + \delta$, either $f(x_1) \leqq f(x_0) \leqq f(x_2)$ or $f(x_1) \geqq f(x_0) \geqq f(x_2)$; a function monotonic at x_0 is not necessarily monotonic in any interval containing x_0. There are then several senses in which a continuous function $f(x)$ may be said to be everywhere oscillating: $f(x)$ may be monotonic in no interval, almost nowhere monotonic (i.e., monotonic at most at the points of a set of measure zero), monotonic at most at the points of a countable set, or monotonic at no point. The most natural questions of the existence of functions monotonic in no interval, belonging to more or less restricted classes, are settled by the functions constructed by P. Köpcke and A. Denjoy,[1] which are monotonic in no interval, and not only absolutely continuous, but differentiable at every point, with bounded derivatives. P. Hartman and R. Kershner[2] have recently given a simple construction of an absolutely continuous function which is monotonic in no interval.

It is evident that an absolutely continuous function cannot be almost nowhere monotonic, since it is surely monotonic at the points of the set where its derivative is not zero. Similarly, it is clear that a function which almost everywhere fails to have a finite derivative is almost nowhere monotonic, since by a well known theorem,[3] such a function will almost everywhere have one of its upper Dini derivatives $+\infty$, and one of its lower Dini derivatives $-\infty$. These considerations tell us nothing about the existence of a continuous function of bounded variation, almost nowhere monotonic; in this note such a function will be constructed. A continuous function $f(x)$ of bounded variation must, however, be monotonic at the points of an uncountable set.[4] For, let the curve $y = f(x)$ $(0 \leqq x \leqq 1)$ have the parametric representation $x = x(s)$, $y = y(s)$ $(0 \leqq s \leqq l, l > 1)$, where s is the arc length. Then[5] $x'(s)^2 + y'(s)^2 = 1$

Received January 16, 1939. The author is a National Research Fellow.

[1] A. Denjoy, *Sur les fonctions dérivées sommables*, Bulletin de la Société Mathématique de France, vol. 43(1915); pp. 161–248; pp. 210 ff. Denjoy gives a critique of Köpcke's construction (pp. 228 ff.).

[2] P. Hartman and R. Kershner, *The structure of monotone functions*, American Journal of Mathematics, vol. 59(1937), pp. 809–822; p. 817.

[3] E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, vol. 1, 1927, p. 400.

[4] I am indebted to A. P. Morse for this remark.

[5] See, e.g., F. Riesz, *Sur l'existence de la dérivée des fonctions monotones et sur quelques problèmes qui s'y rattachent*, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio Scientiarum Mathematicarum [Szeged], vol. 5(1930–32), pp. 208–221; p. 216.