MODULAR FIELDS. I

SEPARATING TRANSCENDENCE BASES

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1. **Introduction.** Any extension K of a given field L has a transcendence basis T over L, that is, a set of elements $T = \{t_1, t_2, \cdots\}$ algebraically independent over L and such that all elements of K are algebraic over T. In other words, K can be considered as a (possibly infinite) field of algebraic functions of the variables $t_1, t_2, \cdots$. Many properties of algebraic equations must be restricted to separable equations, without multiple roots, so we enquire: When does a field K have over a subfield L a "separating" transcendence basis T such that all elements of K are roots of *separable* algebraic equations over T?

Forms of this question arise in the analysis of intersection multiplicities for general algebraic manifolds (B. L. van der Waerden [13][1]), in one method of discussing the structure of complete fields with valuations (Hasse and Schmidt [4], p. 16 and p. 46), and in the study of pure forms over function fields (Albert [2]). The properties of such separating transcendence bases may be also considered as one part of a systematic study of the algebraic structure of fields of characteristic p.

A first result, obtained independently by van der Waerden ([13], Lemma 1, p. 620) and by Albert and the author ([2], Theorem 3), is

Theorem 1. *Any field K obtained by adjoining a finite number of elements to a perfect field P has a separating transcendence basis over P.*

A proof is given in §3 below. The fields treated in this theorem might also be described as finite algebraic function fields of n variables over P, for any integer n. A similar result for a more general ground field is (proof in §7, Theorem 14, Corollary)

Theorem 2. *If L is a function field of one variable over a perfect coefficient field P, and if K is obtained by adjoining to L a finite number of elements in such a way that every element of K algebraic over L is in L, then K has a separating transcendence basis over L.*

The hypothesis that K is generated over L by a finite number of elements is essential to this theorem. For more general fields K there is a relation between the structure of K and that of its subfields over P.

Received January 13, 1939; presented to the American Mathematical Society, December 29, 1938.

[1] Numbers in brackets refer to the bibliography at the end of the paper.