PERSYMMETRIC AND JACOBI DETERMINANT EXPRESSIONS FOR ORTHOGONAL POLYNOMIALS

By Vivian Eberle Spencer

Introduction. Work with orthogonal Tchebycheff polynomials (OP) has usually taken as its point of attack the notion of a weight function and the corresponding moment problem. In the study of OP two important expressions for them as determinants arise. The first, or *persymmetric determinant expression*, is obtained by replacing the elements of the last row of a certain positive persymmetric determinant by powers of x; the second, or *Jacobi determinant expression*, results when the characteristic determinant of a certain Jacobi matrix is written. The present paper undertakes a study of orthogonal and related persymmetric polynomials from the standpoint of the theory of matrices and determinants. In all fundamental theory the notion of a weight function is entirely avoided.

The persymmetric determinant expression leads to a classification of sequences of these polynomials into sets S, each of which is found to contain one and only one symmetric sequence. Properties of sets S are investigated. In considering the Jacobi determinant expression we come upon certain finite sequences[1] $\{\Theta_i(x)\}_1^n$ of orthogonal polynomials which associate themselves in a very simple manner with any sequence of OP $\{\Phi_n(x)\}$. The study of $\{\Theta_i(x)\}_1^n$ leads to new bounds for the zeros of $\Phi_n(x)$. Properties of the continued fraction associated with $\{\Theta_i(x)\}_1^n$ are obtained. Combining these results, we are led to a set of theorems regarding a formally defined interval of orthogonality for $\{\Phi_n(x)\}$. A theorem of Krein[2] has important bearing on our study. We close with its extension. Throughout the paper applications are made to the classical orthogonal polynomials.

For OP we use the notation adopted by J. Shohat.[3]

1. A classification of sequences of persymmetric polynomials.

A *persymmetric or Hankel determinant* is a determinant in which each line perpendicular to the

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[1] By the notation $\{q_i\}_1^n$ will be understood the sequence $\{q_i\}$ $(i = 1, 2, \cdots, n)$.

[2] M. Krein, *Über das Spektrum der Jacobischen Form in Verbindung mit der Theorie der Torsionsschwingungen von Walzen* (in Russian), Rec. Math. Moscou, vol. 40(1933), pp. 455–465.

[3] J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Mémorial des Sciences Math., Fasc. 66(1934).