A GENERALIZED LAMBERT SERIES

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1. **Introduction.** In his important paper of 1913 Knopp [5][1] proposed to call any series of the form

$$\sum_{n=1}^{\infty} b_n x^n (1 - x^n)^{-1} \tag{1.1}$$

a Lambert series in honor of J. H. Lambert who in 1771 was the first to treat special series of this type [6].

The purpose of this paper is to discuss a more general series, namely:

$$\sum_{n=1}^{\infty} b_n x^{\lambda n} h(x^n), \tag{1.2}$$

where λ is a positive integer and $h(x)$ is a function of x which is analytic in the interior of the unit circle and which has a value different from zero at $x = 0$. In §§3 and 4 we suppose further that $h(x)$ has on the unit circle a finite number of singularities of which at least one is a pole.

In his paper Knopp proves the following theorem.[2]

Let the coefficients b_n of the series (1.1) be such that for a definite integer k all the k series

$$\sum_{\nu=1}^{\infty} \frac{b_{k\nu+l}}{k\nu + l} \qquad (l = 0, 1, 2, \cdots, k - 1) \tag{1.3}$$

converge. Then if for such a k and for k' prime to k we write $x_0 = e^{2\pi i k'/k}$, we have for radial approach of x to x_0 the relation

$$\lim_{x \to x_0} \left\{ (1 - x/x_0) \sum_{n=1}^{\infty} b_n x^n (1 - x^n)^{-1} \right\} = \sum_{\nu=1}^{\infty} \frac{b_{k\nu}}{k\nu}.$$

From this theorem it follows that the function defined by the series (1.1) cannot be continued analytically across the unit circle if the hypotheses of the theorem are satisfied for an infinite number of values of k for each of which the series $\sum_{\nu=1}^{\infty} b_{k\nu}/(k\nu)$ satisfies the additional condition of having its sum different from zero.

Recently Mary Cleophas Garvin [2] obtained corresponding results for series of the form

$$\sum_{n=1}^{\infty} b_n x^{\lambda n} (1 - x^{\mu n})^{-1}, \tag{1.4}$$

which are obtained from our series in the case $h(x) = 1/(1 - x^{\mu})$.

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[1] The numbers in brackets refer to the list of references at the end of the paper.

[2] Theorem 3 of §2.