GENERALIZED PROBLEM OF BOLZA IN THE CALCULUS OF VARIATIONS

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1. **Introduction.** The problem to be studied in the present paper is that of minimizing a function

$$(1.1)\qquad I(C) = g(a) + \int_{x_1}^{x_2} f(a, x, y, y')\,dx$$

in a class of admissible arcs C of the form

$$(1.2)\qquad a_h,\quad y_i(x) \qquad (x_1 \leqq x \leqq x_2\,;\, h = 1, \cdots, r;\, i = 1, \cdots, n)$$

in axy-space satisfying the conditions

$$(1.3a)\qquad \varphi_\gamma(a, x, y, y') = 0 \qquad (\gamma = 1, \cdots, m < n),$$

$$(1.3b)\qquad x_s = x_s(a),\quad y_i(x_s) = y_{is}(a) \qquad (s = 1, 2),$$

$$(1.3c)\qquad I_\rho = g_\rho(a) + \int_{x_1}^{x_2} f_\rho(a, x, y, y')\,dx = 0 \qquad (\rho = 1, \cdots, p).$$

The a's are independent of the variable x. In the following pages it will be convenient to designate this problem as *problem* A.

The problem just formulated can be modified in many ways. For example, one can suppose that the functions $x_s(a)$ are constants, since this result can be brought about by replacing x by a new variable t by means of the transformation $x = x_1(a) + t[x_2(a) - x_1(a)]$ $(0 \leqq t \leqq 1)$. Moreover, one can assume that the functions $g(a)$, $g_\rho(a)$ are identically zero, for along an admissible arc C satisfying the conditions (1.3) the function (1.1) can be put in the form

$$I(C) = \int_{x_1}^{x_2} \{f + g/[x_2(a) - x_1(a)]\}\,dx$$

and a similar expression holds for $I_\rho(C)$. The simplicity of these transformations of problem A is due to the presence of the a's in the functions $f, f_\rho, \varphi_\gamma$. The introduction of the a's in these functions not only enlarges the class of problems that are immediate special cases of our problem, but also gives rise to a more symmetric theory. This is particularly true in the theory of Mayer fields, as will be seen in §3 below. However, problem A can be reduced to one in which the functions $f, f_\rho, \varphi_\gamma$ are independent of the a's. This can be done by replacing the variables $a_1, \cdots, a_r$ in these functions by new variables $y_{n+1}(x), \cdots, y_{n+r}(x)$ satisfying the conditions $y'_{n+h} = 0$, $y_{n+h}(x_s) = a_h$ $(s = 1, 2)$. If one

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