CONVERGENCE THEOREMS FOR CONTINUED FRACTIONS

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1. **Introduction.** The purpose of this paper is to present a new set of convergence theorems for continued fractions of the form

$$(1.1) \qquad 1 + \frac{a_1}{1+} \frac{a_2}{1+} \frac{a_3}{1+} \cdots,$$

where the a_n are complex numbers $\neq 0$. The method used is an extension of a method used in an earlier paper (Leighton [1][1]) the results of which now follow from Theorem 4.1 of the present paper.

A number of writers[2] proved independently that if $|a_n| \leqq \frac{1}{4}$ $(n = 2, 3, 4, \cdots)$, the continued fraction (1.1) converges. Szász [1] showed that the constant $\frac{1}{4}$ cannot be improved by proving that the continued fraction

$$\frac{-\frac{1}{4} - e}{1} + \frac{-\frac{1}{4} - e}{1} + \frac{-\frac{1}{4} - e}{1} + \cdots$$

diverges for each value of $e > 0$. Later, new types of sufficient conditions for convergence were found (Leighton and Wall [1], Jordan and Leighton [1], Leighton [2]), but all of these theorems required that at least an infinite subsequence of the $|a_n| \leqq \frac{1}{4}$. This last condition was recently removed (Leighton [1]) by showing that (1.1) converges if

$$(1.1)' \qquad |1 + a_2| \geqq 1 + |a_1|, \qquad |a_2| \geqq \frac{2+m}{1-m},$$

$$|a_{2n+1}| \leqq m < 1, \qquad |a_{2n+2}| \geqq 2 + m + m\,|a_{2n}| \qquad (n = 1, 2, 3, \cdots)$$

It will follow incidentally from Theorem 4.4 of the present paper that this condition can be removed in still different ways.

We recall that the n-th approximant A_n/B_n of a continued fraction

$$(1.2) \qquad \beta_0 + \frac{\alpha_1}{\beta_1+} \frac{\alpha_2}{\beta_2+} \cdots$$

is defined by means of the recursion relations

$$(1.3) \qquad \begin{aligned} &A_0 = \beta_0, \qquad B_0 = 1, \qquad A_1 = \beta_0\beta_1 + \alpha_1, \qquad B_1 = \beta_1, \\ &A_n = \beta_n A_{n-1} + \alpha_n A_{n-2}, \qquad B_n = \beta_n B_{n-1} + \alpha_n B_{n-2}, \quad (n = 2, 3, 4, \cdots). \end{aligned}$$

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[1] Numbers in brackets refer to the bibliography.

[2] For bibliography on this criterion see Szász [1] and Leighton [1].