ASYMPTOTIC FORMS FOR A GENERAL CLASS OF HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS TO THE GENERALIZED LEGENDRE FUNCTIONS

By George E. Albert

1. **Introduction.** The classical differential equation of Jacobi [5][1]

$$(1) \quad (1 - z^2)y'' + \{\beta - \alpha - (\alpha + \beta + 2)z\}y' + \nu(\nu + \alpha + \beta + 1)y = 0$$

is solved by the pair of hypergeometric functions (to be designated as the Jacobi functions)

$$(2) \quad \begin{cases} Y_{\nu,1}^{(\alpha,\beta)}(z) = F(\nu + \alpha + \beta + 1, -\nu; \alpha + 1; \tfrac{1}{2}(1 - z)), \\ Y_{\nu,2}^{(\alpha,\beta)}(z) = [\tfrac{1}{2}(z - 1)]^{-\nu-\alpha-\beta-1} \\ \qquad \cdot F(\nu + \alpha + \beta + 1, \nu + \beta + 1; 2\nu + \alpha + \beta + 2; 2/(1 - z)). \end{cases}$$

In the following pages forms will be derived for the Jacobi functions (2) which are asymptotic with respect to the large parameter ν.

The Legendre functions of complex degree, order, and argument are defined in terms of the Jacobi functions (2) by the formulas

$$(3) \quad \begin{cases} P_\nu^\mu(z) = \dfrac{1}{\Gamma(1 - \mu)} \left(\dfrac{z + 1}{z - 1}\right)^{\frac{1}{2}\mu} Y_{\nu,1}^{(-\mu,\mu)}(z), \\ 2Q_\nu^\mu(z) = e^{\mu\pi i} \dfrac{\Gamma(\nu + 1)\Gamma(\nu + \mu + 1)}{\Gamma(2\nu + 2)} \left(\dfrac{z + 1}{z - 1}\right)^{\frac{1}{2}\mu} Y_{\nu,2}^{(-\mu,\mu)}(z); \end{cases}$$

see Hobson [3] or [4]. In virtue of these relations between the two classes of functions, asymptotic forms will be at hand for the Legendre functions for values of $|\nu|$ which are large in comparison with $|\mu|$, and conversely.

I. The Jacobi functions

2. **The normalization of the differential equation.** In the differential equation (1) the numbers ν, α, and β will be subject to the blanket restrictions that $|\nu|$ be large and $|\alpha|$, $|\beta|$ be bounded; otherwise they are general complex numbers. The variable z will be allowed to range over the unbounded complex plane, cut along the axis of reals from the point $z = 1$ to $z = -\infty$, with the exception of an arbitrarily small neighborhood of the point $z = -1$. The domain of z thus defined will be consistently designated by R_z. In virtue of the known continuation formulas for hypergeometric functions, the omission of any small neighborhood of the point $z = -1$ involves no loss of generality in the results to be obtained.

Received October 8, 1938; in revised form, March 22, 1939.

[1] Numbers in brackets refer to the bibliography at the end of the paper.