DIFFERENTIATION IN BANACH SPACES

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Introduction. Consider a figure[1] R_0 in a Euclidean n-space. According to a classical theorem of Lebesgue, if $\mathfrak{X}$ is the space of real numbers, then every ABV (additive, with bounded variation) function defined to $\mathfrak{X}$ from the figures in R_0 is necessarily differentiable a.e.[2] in R_0. But, as Bochner first pointed out [4], this theorem does not hold for general Banach spaces $\mathfrak{X}$. There exist spaces $\mathfrak{X}$ to which ABV functions may be defined that are differentiable at no point in R_0. Several authors [3, 6, 7, 10, 12, 15] have as a consequence considered the problem of finding conditions on $\mathfrak{X}$ sufficient that every ABV function defined to $\mathfrak{X}$ be differentiable a.e. Here, however, we wish to adopt a somewhat different viewpoint, at least throughout §§1 and 2, the sections fundamental to our discussion: in the principal theorems of the paper, given in §2, the emphasis has been placed on the individual function X_R rather than on the space $\mathfrak{X}$ and the class of all ABV functions having their values in $\mathfrak{X}$. Thus (to put it more explicitly) the conclusions reached in Theorems 2.5, 2.7, 2.8, and 2.9 state, with no restriction on $\mathfrak{X}$, that if a fixed ABV function X_R defined to $\mathfrak{X}$ has a generalized "weak" derivative according to any one of several definitions, then X_R is differentiable a.e.; that is, X_R has a "strong" derivative. In each of these four theorems it is shown that a set of necessary conditions, expressed in terms of linear functionals and apparently quite feeble, are actually of sufficient strength to insure differentiability a.e.

The possible use of these results is not confined to testing the strong differentiability of an individual function having its values in an unrestricted (and perhaps unsatisfactory) space; the theorems can also be applied to the problem considered in the papers cited above, namely, that of testing whether or not a given condition which the space $\mathfrak{X}$ is assumed to satisfy is strong enough to insure the differentiability a.e. of every ABV function defined to $\mathfrak{X}$. The results concerning differentiation that have been obtained in [3], [7], [10], [12], and [15] are here derived in §§3–5 from the theorems of the present §2; in each proof the essential idea is to show that if $\mathfrak{X}$ satisfies the particular condition under consideration, then $\mathfrak{X}$ is weakly compact in one generalized sense or another.

Following §1, in which the necessary definitions have been grouped, the principal theorems will be found in §2. Those dealing with differentiation we

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[1] Terms used, but undefined, in the present paper will be found in either [18] or [1] (numbers in brackets refer to the list of references at the end). It is supposed that the reader is somewhat familiar with these two treatises.

[2] The phrase "almost everywhere" will be abbreviated throughout to "a.e."