A PROOF THAT EVERY UNIFORMLY CONVEX SPACE IS REFLEXIVE

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The purpose of the present note is to communicate an independent proof of a result of Milman's [6][1] to the effect that every uniformly convex space is necessarily reflexive. Our proof is quite different from Milman's, being based on the use of bounded additive measure functions rather than on that of transfinite closure for closed convex sets.

Let $\mathfrak{X} = [x]$ be a Banach space, $\bar{\mathfrak{X}} = [\gamma]$ its adjoint, and $\bar{\bar{\mathfrak{X}}} = [F]$ the adjoint of $\bar{\mathfrak{X}}$. The space $\mathfrak{X}$ is said to be *reflexive*[2] if for each $F_0 \epsilon \bar{\bar{\mathfrak{X}}}$ there is an $x_0 \epsilon \mathfrak{X}$ such that $F_0(\gamma) = \gamma(x_0)$ holds for all $\gamma \epsilon \bar{\mathfrak{X}}$. The concept of $\mathfrak{X}$ being a *uniformly convex* space, a concept of Clarkson's [2], may be defined in the following fashion: *given $\epsilon > 0$ there is a $\zeta_\epsilon > 0$ with the property that*

(*) *if $x, y \epsilon \mathfrak{X}$ have $\| x \| = \| y \| = 1$ and if $\| x - y \| \geqq \epsilon$, then $\| x + y \| \leqq 2 - \zeta_\epsilon$.*

The theorem may now be stated as follows.

Theorem (Milman). *If $\mathfrak{X}$ is isomorphic to a uniformly convex space, then $\mathfrak{X}$ is reflexive.*

We first establish two lemmas.

Lemma 1.[3] *If $\mathfrak{X}$ is uniformly convex, then given $\gamma_0 \epsilon \bar{\mathfrak{X}}$ with $\| \gamma_0 \| \neq 0$, there exists a unique $x_0 \epsilon \mathfrak{X}$ satisfying the conditions $\| x_0 \| = 1$ and $\gamma_0(x_0) = \| \gamma_0 \|$. Moreover, given $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that if x and y in $\mathfrak{X}$ and γ in $\bar{\mathfrak{X}}$ satisfy the conditions $\| x \| = 1$, $\| y \| \leqq 1$, $\| x - y \| \geqq \epsilon$, and $\gamma(x) = \| \gamma \|$, then $\gamma(y) \leqq (1 - \delta_\epsilon) \| \gamma \|$.*

In proving the first part it is clearly sufficient to consider only the case $\| \gamma_0 \| = 1$. By definition of $\| \gamma_0 \|$ there then exists a sequence $\{x_n\}$ with $\| x_n \| = 1$ and $1 \geqq \gamma_0(x_n) > 1 - n^{-1}$. To see that $\{x_n\}$ is a Cauchy sequence

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[1] Numbers in brackets refer to the list of references.

[2] Such spaces were introduced by Hahn [4] under the name of *regular*. The present term *reflexive* is due to Lorch.

[3] The first statement contained in Lemma 1 was discovered independently by J. A. Clarkson and E. J. McShane in 1936. I am grateful to them for permission to include it here.