ON IMBEDDING A SPACE IN A COMPLETE SPACE

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If S is a metric space, a sequence $p_k \epsilon S$ is a Cauchy sequence if for every $\epsilon > 0$ there is a k_ϵ such that, for $k \geqq k_\epsilon$, $h > 0$, $\rho(p_k, p_{k+h}) < \epsilon$. If $S(p, \epsilon)$ is the sphere of radius ϵ, this condition may be expressed as

$$p_k \epsilon S(p_{k_\epsilon}, \epsilon), \qquad k \geqq k_\epsilon.$$

It is a classical result that for any metric S there exists a complete metric S^* such that S is homeomorphic to a subset of S^* and the image of a Cauchy sequence in S is a convergent sequence in S^*. If we consider a class S of elements p, called points, and a class A of elements α, called indices, such that for each $p \epsilon S$ and $\alpha \epsilon A$ there is a well defined subset $U_\alpha(p) \subset S$, we may study an analogous problem. A sequence $p_k \epsilon S$ may be called a Cauchy sequence if for every $\alpha \epsilon A$ there are a $q_\alpha \epsilon S$ and a $k_\alpha > 0$ such that

$$p_k \epsilon U_\alpha(q_\alpha), \qquad k \geqq k_\alpha.$$

The problem is that of formulating conditions upon the $U_\alpha(p)$ such that a theorem analogous to the one given above for metric S may be proved. We shall give a solution of this problem in this paper. The method is related to that used by Cantor in defining the real numbers as classes of equivalent sequences of rational numbers. The conditions are related to those of Chittenden, Alexandroff and Urysohn, Niemytski and Weil in their work on the metrization problem and uniformity properties of topological spaces.

The space S, the index set A and the sets $U_\alpha(p)$, called neighborhoods, are subjected to the following postulates:

I. $\prod_{\alpha \epsilon A} U_\alpha(p) = p$.

II. If $p \epsilon S$ and $\alpha, \beta \epsilon A$, there is $\gamma = \gamma(\alpha, \beta; p)$ such that $U_\alpha(p)U_\beta(p) \supset U_\gamma(p)$.

III. If $p \epsilon S$ and $\alpha \epsilon A$, then there are $\lambda(\alpha)$, $\delta(p, \alpha) \epsilon A$ such that, if $q \epsilon S$ and

$$U_{\delta(p,\alpha)}(q)U_{\lambda(\alpha)}(p) \neq 0,$$

then

$$U_{\delta(p,\alpha)}(q) \subset U_\alpha(p).$$

We shall refer to $\lambda(\alpha)$ as the first index of III and $\delta(p, \alpha)$ as the second index of III.

If $M \subset S$, $\bar{M}$ is the set of all $p \epsilon S$ such that, for all $\alpha \epsilon A$, $U_\alpha(p)M \neq 0$.

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