A GENERALIZATION OF PICARD'S AND RELATED THEOREMS

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With the exception of the generalization of Picard's second theorem, the results contained in this paper are already known. But new (and quite simple) proofs of the various theorems to which the title refers are given.[1] The necessary preliminary material has also been included, so that only the elements of function theory are assumed as known. In particular, the theorems with whose generalizations we are concerned are not assumed, but are obtained as special cases of our results.

1. **Introduction.** An analytic function $f(z)$ is said to be meromorphic in a given region if it has no singularities there except poles. And indeed, in considering meromorphic functions, a pole is not to be regarded as a singularity, but rather simply as a point where the value of the function is ∞. The function which is equal to ∞ throughout the given region is also to be regarded as meromorphic there. By the expression "meromorphic function" we shall mean a function which is meromorphic for all values of z except $z = \infty$. (Since a function which is meromorphic for all values of z is a rational function, no new term is needed in that case.)

We shall call a number a an excluded value for $f(z)$ if the equation $f(z) = a$ has no root in the given region. Picard's first theorem may then be stated as follows: *A meromorphic function which admits more than two excluded values is a constant.*

We shall say that a is an exceptional value of order m if the equation $f(z) = a$ has no root of multiplicity less than m. The symbol m in that statement represents a positive integer or ∞. For $m = 1$ no condition is imposed on the function, while for $m = \infty$ the statement says that $f(z) = a$ has no root of any order. We wish now to assign weights to the exceptional values; we shall say, so to speak, that an exceptional value of order m has an importance which is a certain fraction of that of an excluded value. From what we have said, it is clear that the weight should increase with m and that it should be 0 for $m = 1$ and 1 for $m = \infty$. The expression $1 - 1/m$ suggests itself for the weight; we shall take this as the definition, since it proves useful. (This idea of weight

[1] The previously known proofs depend on Nevanlinna's theory of meromorphic functions; see his book, *Le Théorème de Picard-Borel*, 1929, p. 102. For a discussion of the results from a different point of view, see Ahlfors, *Quelques propriétés des surfaces de Riemann*, Bull. de la Soc. Math. de France, vol. 60(1932), pp. 197–207. The generalized criterion for a normal family of meromorphic functions seems to have been stated first by Bloch (making use of Nevanlinna's results); see, e.g., Valiron, *Familles Normales de Fonctions Méromorphes*, 1929, p. 21.