CERTAIN DIFFERENTIAL EQUATIONS FOR TCHEBYCHEFF POLYNOMIALS

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1. **Introduction.** The classical orthogonal polynomials of Jacobi, Laguerre, and Hermite satisfy a differential equation of the form

$$(l_{22}x^2 + l_{21}x + l_{20})y_n''(x) + (l_{11}x + l_{10})y_n'(x) + l_{00}y_n(x) = \lambda_n y_n(x),$$

where the $\{l_{ij}\}$ are constants and λ_n is a parameter. By repeated iterations of this equation one can obtain other differential equations of higher order which have these orthogonal polynomials as solutions. For example, the Legendre polynomials satisfy

$$(x^2 - 1)y_n''(x) + 2xy_n'(x) = n(n + 1)y_n(x),$$

$$(x^2 - 1)^2 y_n^{\mathrm{iv}}(x) + 8x(x^2 - 1)y_n'''(x) + (14x^2 - 6)y_n''(x) + 4xy_n'(x) = n^2(n + 1)^2 y_n(x).$$

However, all the iterates have a special form, namely, the coefficient of the r-th derivative is a polynomial of degree $\leqq r$.

In this paper we shall look for polynomial solutions, in particular, for orthogonal polynomial solutions, of the general differential equation of this type:[1]

$$L(y) = \sum_{i=0}^{r}\left(\sum_{j=0}^{i} l_{ij}x^j\right)y_n^{(i)}(x) = \lambda_n y_n(x), \tag{1}$$

where

$$\lambda_n = l_{00} + nl_{11} + n(n - 1)l_{22} + \cdots .$$

We also consider an extended definition of orthogonal polynomials which we call a Tchebycheff set.

Definition. *Given a set of real or complex constants* $\{c_n\}$ *such that*

$$\Delta_{nn} = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-2} \end{vmatrix} \neq 0 \qquad (n = 1, 2, \cdots), \tag{2}$$

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[1] It is obvious that there would be no loss of generality in assuming that $l_{00} = 0$, for this term can be absorbed in the λ_n .