PROOF OF A GAP THEOREM

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Using the theory of Fourier transforms, Wiener[1] proved the following

Theorem. *Let us suppose that the trigonometrical series*

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos \lambda_k x + b_k \sin \lambda_k x), \tag{1}$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, *is quadratically bounded over an interval* (a, b), *that is, that there exists a number* M *such that*

$$\int_a^b s_n^2(x)\, dx \leqq M^2, \tag{2}$$

where

$$s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos \lambda_k x + b_k \sin \lambda_k x)$$

for $n = 1, 2, \cdots$.[2] *Let*

$$\lambda_n - \lambda_{n-1} \geqq \Delta > 0 \qquad (n = 1, 2, \cdots). \tag{3}$$

We write $b - a = \delta$. *Then, if* Δ *is sufficiently large,*

$$\Delta \geqq \Delta_0 = \Delta_0(\delta), \tag{4}$$

the series $\sum (a_k^2 + b_k^2)$ *converges, and*

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leqq A(\delta) M^2, \tag{5}$$

where $A(\delta)$ *is a constant depending only on* δ.[3]

The object of this note is to give a new and elementary proof of this theorem.

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[1] N. Wiener, *A class of gap theorems*, Annali di Pisa, vol. 2(1934), pp. 367–372.

[2] We may restrict ourselves to the case of real coefficients.

[3] The numbers λ_i need not be integers. If $\lambda_1, \lambda_2, \cdots$ are integers, the theorem may also be stated as follows:

Under the conditions (2), (3), (4), *the series* (1) *converges in mean to a function* $f(x)$ *such that*

$$\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2\, dx \leqq A(\delta) \int_a^b |f(x)|^2\, dx.$$