THE INDEX THEOREM IN THE CALCULUS OF VARIATIONS

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Introduction. The calculus of variations in the large is concerned with boundary problems in the large. Of these problems the simplest is that of finding extremals joining two points A and B on a regular m-manifold M. The theory[1] obtains relations between the local characteristics of the solutions of the problem and the topological characteristics (connectivities, etc.) of the space of admissible curves. The case where A and B are not conjugate on any extremal solution is termed the *non-degenerate case*. This case is the general case in the sense that for A fixed the set of points B which are conjugate to A on at least one extremal issuing from A has a null m-dimensional measure on M. The unrestricted case can be treated as a limiting case of the non-degenerate case (M, p. 239).

It appears that the most significant characteristic of an extremal solution g in the non-degenerate case is the number μ of conjugate points of A on g. This fact becomes most evident in terms of the "Index Theorem". Recall that the "index" of a critical point $(z) = (0)$ of a function $J(z)$ of a finite number of variables (z) is the number of negative characteristic roots of the Hessian of $J(z)$ at the point $(z) = (0)$. As we shall see $J(z)$ will represent the value of the integral J along a "canonical" broken extremal neighboring g with vertices determined by (z). The Index Theorem affirms that μ equals the index of the critical point (0) of this function $J(z)$. It is by means of this theorem that the topological characteristics of the neighborhood of g among admissible curves are determined.

The Index Theorem was first established in the non-parametric case in 1929 by Morse.[2] It was established in the parametric case by a reduction to the non-parametric case (M, p. 138). Recently the author has discovered a new and simpler method of proving the theorem. This method can be applied

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[1] M. Morse, *The Calculus of Variations in the Large*, American Mathematical Society Colloquium Publications, vol. 18, New York, 1934. A reference to these lectures will be indicated by the letter M.

Classical treatments of the conjugate point condition can be found in the following references.

G. A. Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 195–206.

C. Carathéodory, *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*, Berlin, Teubner, 1935.

[2] M. Morse, *The foundations of the calculus of variations in the large in m-space*, Transactions of the American Mathematical Society, vol. 31 (1929), pp. 379–404.