TAUBERIAN THEOREMS FOR $(C, 1)$ SUMMABILITY

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Let $\sum_{n=1}^{\infty} u_n$ be an infinite series, with partial sums $s_n = u_1 + u_2 + \cdots + u_n$.[1] It is well known that the applicability of Cesàro summation to the series is limited in various ways. On the one hand, the u_n cannot be too large if the series is to be summable at all;[2] on the other hand, it was shown by G. H. Hardy that if the u_n are too small, the series cannot be summable without being convergent.[3] The object of this note is to point out that in addition the Cesàro means of the series cannot approach a limit very rapidly unless the series is convergent. For simplicity, we restrict ourselves to $(C, 1)$ summability; we write $\sigma_n = n^{-1}(s_1 + s_2 + \cdots + s_n)$; then the given series is summable $(C, 1)$ to s if $\lim_{n\to\infty} \sigma_n = s$. Our theorem is

THEOREM 1. *If, as* $n \to \infty$, $\sigma_n - s = o(n^{-\epsilon})$ $(0 \leqq \epsilon < 1)$ *and* $u_n < O(n^{\epsilon-1})$, *then* $s_n \to s$; *if* $\sigma_n - s = O(n^{-\epsilon})$ $(0 < \epsilon \leqq 1)$ *and* $u_n < o(n^{\epsilon-1})$, *then* $s_n \to s$.

For $\epsilon = 0$, we have the known one-sided generalization of Hardy's theorem. The first part of Theorem 1 would be trivial for $\epsilon = 1$; the second part would be false for $\epsilon = 0$.

The integral analogue of Theorem 1 is

THEOREM 2. *If $g(t)$ is the derivative of its integral on every finite interval* $(0, x)$, *and if, as* $x \to \infty$,

$$\int_0^x (1 - x^{-1}t)g(t)\,dt - s = o(x^{-\epsilon}) \qquad (0 \leqq \epsilon < 1)$$

and $g(x) < O(x^{\epsilon-1})$; *or if*

$$\int_0^x (1 - x^{-1}t)g(t)\,dt - s = O(x^{-\epsilon}) \qquad (0 < \epsilon \leqq 1)$$

and $g(x) < o(x^{\epsilon-1})$; *then* $\int_0^\infty g(t)dt = s$.

The case $\epsilon = 0$ is known.[4]

Received January 12, 1938. The author is a National Research Fellow.

[1] All numbers in this note are real.

[2] If the series is summable (C, r) $(r > -1)$, $u_n = o(n^r)$ $(n \to \infty)$. See, for example, E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, vol. 2, 1926, p. 77.

[3] See E. W. Hobson, op. cit., p. 81.

[4] See E. W. Hobson, op. cit., p. 388.