ON ABSOLUTELY CONTINUOUS TRANSFORMATIONS IN THE PLANE

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1. Introduction

1.1. We shall be concerned with continuous transformations T given in the form

$$T\colon x = x(u, v), \qquad y = y(u, v),$$

where $x(u, v)$, $y(u, v)$ are continuous in the closed *fundamental square* $S_0\colon 0 \leqq u \leqq 1, 0 \leqq v \leqq 1$. *We do not assume that T is bi-unique.* Let us use K to denote any finite system of closed squares s in S_0, without common interior points. Let $\bar{s}$ denote the image of s and $|\bar{s}|$ the measure[1] of $\bar{s}$. If there exists a finite constant M such that

$$\sum_{s \in K} |\bar{s}| < M$$

for every system K, then T is of *bounded variation in the sense of Banach.* If for every $\epsilon > 0$ there exists an $\eta = \eta(\epsilon) > 0$ such that

$$\sum_{s \in K} |\bar{s}| < \epsilon$$

for every system K which satisfies the condition

$$\sum_{s \in K} |s| < \eta(\epsilon),$$

then T is *absolutely continuous in the sense of Banach* (Banach [2]).

1.2. A real function $f(u)$ of the real variable u, continuous in an interval $a \leqq u \leqq b$, gives rise to a one-dimensional continuous transformation $x = f(u)$. The theory of the transformations T, defined in §1.1, appears thus as a two-dimensional generalization of the theory of functions of a single variable which are of bounded variation and absolutely continuous, respectively. In the one-dimensional case, we know that if $f(u)$ is of bounded variation, then $f'(u)$ exists almost everywhere in the interval $a \leqq u \leqq b$ and is summable there. Furthermore, if $f(u)$ is absolutely continuous, then we have the fundamental identities

$$\int_a^b f'(u)du = f(b) - f(a), \tag{1}$$

$$\int_a^b |f'(u)|\, du = V(f; a, b), \tag{2}$$

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[1] If E is a measurable set, then $|E|$ denotes the measure of E.