TRANSFORMATIONS IN LINEAR TOPOLOGICAL SPACES

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1. **Introduction.** In this paper some well known theorems for linear metric spaces whose proofs do not involve completeness are extended to linear topological spaces. The relation between bounded sets and linear continuous transformations is considered; and, in connection with this, metrizability conditions for linear topological spaces are obtained in terms of boundedness. A linear continuous functional on a linear topological space is shown to operate essentially on only a "part" of the space which can be normed. Finally the relation between completeness and category is considered and a theorem is proved which shows that two definitions of completeness which have been given do not imply that the space is of the second category.

2. **Topologies for linear spaces.** In the postulates for a linear set [cf. **1**, p. 26][1] there is no notion of a topology. If a topology is imposed on a linear set in such a fashion that the postulated operations of addition of elements and multiplication of elements by real numbers are continuous in the topology, the linear set will be said to be a linear topological space [cf. **2**, pp. 201–204]. The most important topologies of this type for linear sets have been metric topologies, viz., the F-metric [**1**, p. 35] and the norm [**1**, p. 53].[2] As a matter of fact, it will be shown that the F-metric is the most general metric of this nature.

However, two non-metric topologies have been given in the literature. Kolmogoroff [**3**, p. 29] has simply postulated an operation of closure, $\bar{M}$, for any set M satisfying the Riesz-Kuratowski axioms, viz.,

(1) if M is a single element, $\bar{M} = M$,

(2) $\bar{\bar{M}} = \bar{M}$,

(3) $\overline{M + N} = \bar{M} + \bar{N}$,

and then in addition required that $x + y$ and αx be continuous in the topology. He proves that such a space is a regular Hausdorff space. On the other hand, John von Neumann [**4**, p. 4] has defined a topology for a linear set L by means of a set of neighborhoods $\mathfrak{U} = \{U\}$ satisfying the following axioms:

(1) $\theta \in U$ for every $U \in \mathfrak{U}$.

(2) If $U, V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $W \subset \mathfrak{P}(U, V)$.

(3) If $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V + V \subset U$.

(4) If $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $\alpha V \subset U$ for $| \alpha | \leqq 1$.

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[1] Boldface numbers refer to the bibliography at the end of the paper.

[2] Other non-continuous metrics have been introduced by C. R. Adams [**12**, pp. 422, 423].