SPHEROIDAL AND BIPOLAR COÖRDINATES

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1. **The relations between the different coördinates.** Let $x = w \cos \phi$, $y = w \sin \phi$, then if

$$(1.1) \qquad z = r \cos \theta = kuv = kS \operatorname{sh} \sigma, \qquad u \geqq 1, \qquad -1 \leqq v \leqq 1,$$

$$(1.2) \qquad w = r \sin \theta = k(u^2 - 1)^{\frac{1}{2}}(1 - v^2)^{\frac{1}{2}} = kS \sin \tau,$$

$$(1.3) \qquad k(u - v) = R, \qquad k(u + v) = R', \qquad S(\operatorname{ch} \sigma - \cos \tau) = 1,$$

$$(1.4) \qquad (u - v)e^{\sigma} = u + v, \qquad (u^2 - v^2)\cos \tau = u^2 + v^2 - 2,$$

$$(1.5) \qquad r^2 = k^2(u^2 + v^2 - 1).$$

It is usual to call (r, θ, ϕ) the spherical polar coördinates, (z, w, ϕ) the cylindrical coördinates, (u, v, ϕ) the spheroidal coördinates and (σ, τ, ϕ) the bipolar coördinates of the point P whose rectangular coördinates are (x, y, z).

For a second point P_0 whose rectangular coördinates are (x_0, y_0, z_0), quantities u_0, v_0, w_0, θ_0, ϕ_0, σ_0, τ_0, R_0, R_0', S_0 may be defined by similar equations with a constant k_0 which may or may not be different from k. We shall, however, be interested in a function $G(x, y, z, x_0, y_0, z_0)$ which is harmonic when considered as a function of x, y, z and also when considered as a function of x_0, y_0, z_0. For reasons of symmetry it will be convenient in this case to take $k_0 = k$.

2. **The standard spheroidal harmonics.** It is well known that Laplace's equation has the simple solutions

$$P_n^m(u)P_n^m(v)e^{im\phi}, \qquad Q_n^m(u)P_n^m(v)e^{im\phi},$$

where $P_n^m(u)$ and $Q_n^m(u)$ are associated Legendre functions.

In the case of symmetry about the axis of z the simple solutions become

$$P_n(u)P_n(v) \quad \text{and} \quad Q_n(u)P_n(v).$$

A series of solutions of the second type is particularly useful for the representation of a potential function in the space outside a prolate spheroid whose foci are at the points with rectangular coördinates $(0, 0, k)$, $(0, 0, -k)$, respectively. This leads to the consideration of Neumann series of type

$$(2.1) \qquad f(u) = \sum_{n=0}^{\infty} (2n + 1)c_n Q_n(u).$$

Such a series is known to converge in the region of the complex u-plane that lies outside an ellipse with the points $+k$ and $-k$ as foci, when $f(u)$ is an analytic

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