ON PERFECT METHODS OF SUMMABILITY

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1. **Introduction.** In this paper we are concerned exclusively with Toeplitz methods of summability in the real domain, and we begin by introducing the definitions and notations which we shall employ. Being given a matrix $A = (a_{nk})$ $(k, n = 0, 1, 2, \cdots)$ and a sequence $x = \{s_k\}$, we may form the new sequence $y \equiv A(x) \equiv \{t_n\}$ provided each of the series $\sum_{k=0}^{\infty} a_{nk} s_k \equiv t_n \equiv A_n(x)$ is convergent. If y belongs to the space (c) of convergent sequences, we say that x is *summable by the method* A, or simply A-*summable*, and we write A-lim $x =$ lim y. The class $[A]$ of all A-summable sequences is called the *convergence-field* of A. If for two methods A and B we have the relation $[A] \subset [B]$, we say that B *is not weaker than* A. A and B are said to be *consistent* if A-lim $x =$ B-lim x whenever these limits exist. The method I defined by the matrix (δ_{nk}), where δ_{nk} is Kronecker's symbol, is called the *identical method* or the *identity*; obviously $[I] = (c)$. Every method A for which $[I] \subset [A]$ is called *convergence-preserving*; if, in addition, A is consistent with I, it is said to be *regular*. If the matrix (a_{nk}) is such that $a_{nk} = 0$ for $k > n$, A is said to be *triangular*; if, furthermore, $a_{nn} \neq 0$ for every n, A is said to be *normal*. A will be called *reversible* if the equation $A(x) = y$ has exactly one solution x, convergent or not, for each y in (c). For triangular methods the notions of reversibility and normality are easily seen to be equivalent.

For future reference we list here the following conditions which are necessary and sufficient for A to be regular:

$$\lim_{n \to \infty} a_{nk} = 0 \qquad (k = 0, 1, 2, \cdots), \tag{1.1}$$

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1, \tag{1.2}$$

$$\sum_{k=0}^{\infty} | a_{nk} | \leqq K \qquad (n = 0, 1, 2, \cdots). \tag{1.3}$$

We shall say[1] that A is of *type M* if the conditions

$$\sum_{n=0}^{\infty} | \alpha_n | < \infty, \qquad \sum_{n=0}^{\infty} \alpha_n a_{nk} = 0 \qquad (k = 0, 1, 2, \cdots) \tag{1.4}$$

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[1] Matrices of this type were first introduced by Mazur in connection with normal methods; see *Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toeplitzschen Limitierungsverfahren,* Studia Mathematica, vol. 2 (1930), pp. 40–50. We shall refer to this paper hereafter as SM.