THE DEGREES OF THE IRREDUCIBLE COMPONENTS OF SIMPLY TRANSITIVE PERMUTATION GROUPS

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1. In a study of certain hyperorthogonal groups,[1] there arose the problem of splitting into its r irreducible components a simply transitive permutation group G^* of degree n and order g, which, when written in matrix form, gave an isomorphic representation of the given abstract group as a group G of linear transformations. In any simply transitive permutation group, the subgroup leaving one symbol invariant will permute the remaining symbols in $\lambda = r - 1$ sets of transitivity of $k_1, k_2, \cdots, k_\lambda$ symbols respectively. Let the distinct irreducible components of the group G have the degrees $n_0 = 1, n_1, \cdots, n_{\lambda'}$, and note that $r = \lambda + 1$ is the sum of the squares of the multiplicities with which these occur in the reduction of G.[2] When the components are all distinct, and $\lambda' = \lambda$, there appears to be a simple relation between the product $K = 1 \cdot k_1 k_2 \cdots k_\lambda$ and the product $N = 1 \cdot n_1 n_2 \cdots n_\lambda$.

Conjectured Theorem I. *$n^{\lambda-1}K/N$ is an integer when the components of G are distinct, and this is a perfect square R^2 when the numbers k_i are distinct.*

We shall prove the theorem for all groups for which $\lambda \leqq 3$, (here λ' must equal λ), and for an infinite family of groups including all values of λ. When $\lambda = 1$, the group G^* is doubly transitive and $n_1 = k_1 = N = K = n - 1$, so the result is trivial. When $\lambda = 2$, our theorem gives us a diophantine equation, $nk_1k_2/n_1n_2 = R^2$, which, with $n_1 + n_2 = n - 1$, enables us to solve for the unknowns n_1 and n_2.

To illustrate the application of the theorem, before passing to the details of the proof, we take as an example the case of the hyperorthogonal groups, where the problem of this paper was suggested.[1] We have here a permutation group of degree Q_mQ_{m-1}/Q_2 (where $Q_m = q^m - (-1)^m$, $q = p^s$, p prime) which is known to have 3 irreducible components. We know also that $k_1 = q^{2m-3}$, $k_2 = q^2Q_{m-2}Q_{m-3}/Q_2$. Hence, $n_1n_2 = q^{2m-1}Q_mQ_{m-1}Q_{m-2}Q_{m-3}/Q_2^2R^2$, where R is an integer, and $n_1 + n_2 = n - 1 = (Q_mQ_{m-1} - Q_2)/Q_2$. The degree of n_1 or n_2 as a polynomial in q is $2m - 3$, that of the other being less; so the degree of R in q is at least $m - 2$. Since n_1n_2 is divisible by an odd power of q, and $n_1 + n_2$ is divisible by q^2, it follows that n_1, say, is divisible by q^2 but not q^3, and n_2 by q^3 or some higher odd power. R, not being divisible by q^{m-2}, must contain a factor

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[1] J. S. Frame, *Unitäre Matrizen in Galoisfeldern*, Commentarii Mathematici Helvetici, vol. 7 (1935), pp. 97, 98.

J. S. Frame, *The simple group of order 25920*, this Journal, vol. 2 (1936), p. 477.

[2] W. Burnside, *On the complete reduction of any transitive permutation group*, Proc. Lond. Math. Soc., ser. 2, vol. 3 (1905), p. 239.