TOPOLOGICAL FOUNDATIONS IN THE THEORY OF CONTINUOUS TRANSFORMATION GROUPS

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1. Introduction. This paper contains an account of some elementary topology connected with the notion of continuous group of transformations. From the modern point of view the group structures which are studied in the classical Lie theory are frequently not groups at all and have group-like properties only in restricted regions[1] of definition. This circumstance is due partly to the nature of the analysis involved and partly to the topological inadequacy of the ordinary types of coördinate systems and is therefore unavoidable. As a result, the group concepts of the classical theory are necessarily rather nebulous. We have attempted, however, to bring these concepts into sharper being, to crystallize their topological properties by means of definitions formulated postulationally in the spirit of the modern theory. We have not tried to obtain the highest degree of generality, or abstraction; our object is rather to define as simply as possible the types of group structures that one actually encounters and to study some of their simplest topological properties. We shall appeal occasionally to results in the Lie theory, but for the most part we have made it a point to treat situations which are essentially topological with topological methods.[2]

After the preliminary definitions we consider the question whether partial structures are essentially more general than total or completely defined structures or whether on the contrary every partial structure can be considered as being simply a piece of some total structure. In the most general case the answer is as yet unknown. We have, however, settled the question in certain special cases. In this connection we require a preliminary examination of the notion of transitivity, particularly transitivity in the neighborhood of a point, and this in turn requires the study of the topology of spaces whose elements are cosets of a given partial subgroup. In the final section we obtain new relations between the fundamental group of a given continuous group $\mathfrak{G}$ and that of a space $\mathfrak{X}$ in which $\mathfrak{G}$ operates transitively. Some results of this sort have already been obtained by Cartan;[3] our relations, however, have a more quantitative character, since they involve the ranks of certain subgroups of the fundamental

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[1] For this reason expositions of the Lie theory which do not define constantly the regions in which the objects dealt with have their existence would appear to have little validity beyond a purely formal analytic one. This systematic preoccupation with domains of definition seems only to occur in the classic treatise of Lie and Engel [9].

[2] In the study of Lie groups the advantages which are gained from an interplay between topology and analysis are excellently illustrated in Cartan's monograph [2]. We shall draw frequently from the ideas suggested in this work.

[3] [2], p. 27. See also Ehresmann [3], p. 399.