PROOF THAT EVERY POSITIVE INTEGER IS A SUM OF FOUR INTEGRAL SQUARES

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The proof here given of the named classical theorem is a little longer than that offered by L. E. Dickson[1] in 1924, but has some elements of interest on account of the elegance of the method. Moreover, the reciprocal relations employed are of interest in themselves. Of the known proofs ours is most closely related to those of Dickson and Euler.

Let us write[2]

$$(1) \qquad a^2 + \alpha b^2 + \beta c^2 + \alpha\beta d^2 = pq,$$

where $a, b, c, d, \alpha, \beta, p, q$ are integers, p, q are positive and α, β are not negative. Let $x, y, z, t, \lambda, \mu, \rho, \sigma$ be integers, and write

$$(2) \qquad \begin{aligned} Aq = &\{q\lambda + (ax - \alpha by - \beta cz - \alpha\beta dt)\}^2 + \alpha\{q\mu + (bx + ay + \beta dz - \beta ct)\}^2 \\ &+ \beta\{q\rho + (cx - \alpha dy + az + \alpha bt)\}^2 + \alpha\beta\{q\sigma + (dx + cy - bz + at)\}^2. \end{aligned}$$

The sum of the squares of the parenthesis quantities within the braces, multiplied by the indicated outside factors, is

$$(3) \qquad (a^2 + \alpha b^2 + \beta c^2 + \alpha\beta d^2)(x^2 + \alpha y^2 + \beta z^2 + \alpha\beta t^2),$$

or $pq(x^2 + \alpha y^2 + \beta z^2 + \alpha\beta t^2)$, in accordance with the usual (and readily verified) product theorem for the forms in question. Hence A is an integer, and we have

$$(4) \qquad \begin{aligned} A = &\ q(\lambda^2 + \alpha\mu^2 + \beta\rho^2 + \alpha\beta\sigma^2) + p(x^2 + \alpha y^2 + \beta z^2 + \alpha\beta t^2) \\ &+ 2a(\lambda x + \alpha\mu y + \beta\rho z + \alpha\beta\sigma t) + 2\alpha b(-\lambda y + \mu x + \beta\rho t - \beta\sigma z) \\ &+ 2\beta c(-\lambda z - \alpha\mu t + \rho x + \alpha\sigma y) + 2\alpha\beta d(-\lambda t + \mu z - \rho y + \sigma x). \end{aligned}$$

The expression for A is invariant under the transformation

$$(5) \qquad (p, q)(x, \lambda)(y, \mu)(z, \rho)(t, \sigma)(a, a)(b, -b)(c, -c)(d, -d)(\alpha, \alpha)(\beta, \beta).$$

So is equation (1). Hence we may perform on (2) the transformation (5) to obtain the relation

$$(6) \qquad \begin{aligned} Ap = &\ (px + a\lambda + \alpha b\mu + \beta c\rho + \alpha\beta d\sigma)^2 + \alpha(py - b\lambda + a\mu - \beta d\rho + \beta c\sigma)^2 \\ &+ \beta(pz - c\lambda + \alpha d\mu + a\rho - \alpha b\sigma)^2 + \alpha\beta(pt - d\lambda - c\mu + b\rho + a\sigma)^2. \end{aligned}$$

It may also be verified directly that (6) is implied by (1) and (4). In the presence of (1) equations (2) and (6) are equivalent. They constitute the *reciprocal relations* on which our proof is based.

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[1] L. E. Dickson, Amer. Journ. of Math., vol. 46 (1924), pp. 1–16; see esp. pp. 2–5.

[2] We here need the immediately following results (two paragraphs) only for $\alpha = \beta = 1$, but it seems well to put the more general formulas on record.