ORTHOGONALITY IN LINEAR METRIC SPACES

By Garrett Birkhoff

1. **Statement of main theorem.** Let B be any linear metric space[1] of three dimensions, whose points we shall suppose mapped linearly onto those of ordinary space.

It is natural to call a vector $\overline{pq}$ issuing from a point p of B "perpendicular" to a second such vector $\overline{pr}$ [in symbols, $\overline{pq}\perp\overline{pr}$] if and only if there is no point on the extended line through $\overline{pr}$ nearer to q than p.

Remark. Since translations of space are isometric, and uniform expansions about the origin multiply all distances by a constant factor of proportionality, $\overline{pq}\perp\overline{pr}$ implies that any vector parallel or anti-parallel to $\overline{pq}$ is perpendicular to any vector issuing from the same point and parallel or antiparallel to $\overline{pr}$. Therefore it is legitimate to say that the *direction* of $\overline{pq}$ is perpendicular to the *direction* of $\overline{pr}$.

The main purpose of this paper is to prove

Theorem 1. *If $\overline{pq}\perp\overline{pr}$ implies $\overline{pr}\perp\overline{pq}$, and if there is at most one perpendicular from a given line to a point not on that line, then B is "equivalent" to cartesian space (i.e., isometric with it under a linear transformation).*

2. **Outline of proof.** The proof of Theorem 1 involves such simple ideas that it is sufficient to sketch it.

First, let us fix on a particular linear representation of B in ordinary space. It is clear that the metric of B is determined by the "unit pseudo-sphere" S of points whose absolute values (in the terminology of von Neumann) are unity. It is also clear that S is a convex surface.

The argument then proceeds in two main steps. First it is shown that relative to any choice of cylindrical coördinates, the equation defining S is of the form

$$r = f(z)\cdot g(\theta). \tag{1}$$

Then it is shown (in effect) that any plane section of such a surface is an ellipse, essentially completing the proof.

To establish equation (1), let us first note that the radius $\overline{os}$ from the origin o to any point s on S is perpendicular to every line in any plane of support of S at s. Hence by the uniqueness and reciprocity of perpendicularity, S can have at most one plane of support at s.

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[1] As defined for instance by J. von Neumann, *On complete topological spaces*, Trans. Am. Math. Soc., vol. 37 (1935), pp. 3–4. The reader's attention is called to the definition of orthogonality in B. D. Roberts' *On the geometry of abstract vector spaces*, Tôhoku Math. Jour., vol. 39 (1934), pp. 42–59, which is essentially different.