SURFACES WITH PRESCRIBED GAUSS CURVATURE

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1. Introduction. Let $S_g = (\mathbb{R}^2, g)$ denote a conformally flat surface over $\mathbb{R}^2$ with metric given by

$$ds^2 = g^{ij}\, dx_i\, dx_j = e^{2u(x)}\big(dx_1{}^2 + dx_2{}^2\big), \tag{1.1}$$

where u is a real-valued function of the isothermal coordinates $x = (x_1, x_2) \in \mathbb{R}^2$. If u is given, the Gauss curvature function K for S_g is then explicitly given by

$$K(x) = -e^{-2u(x)}\Delta u(x), \tag{1.2}$$

where Δ is the Laplacian for the standard metric on $\mathbb{R}^2$. The quantity

$$\mathcal{K}(u) \equiv \int_{\mathbb{R}^2} K(x)e^{2u(x)}\, dx, \tag{1.3}$$

where dx denotes Lebesgue measure on $\mathbb{R}^2$, is called the integral curvature of the surface (sometimes called total curvature). We say that S_g is a *classical* surface over $\mathbb{R}^2$ if $u \in C^2(\mathbb{R}^2)$. Clearly, $K \in C^0(\mathbb{R}^2)$ in that case. The inverse problem, namely, to prescribe K and to find a surface S_g pointwise conformal to $\mathbb{R}^2$ for which K is the Gauss curvature, renders (1.2) a semilinear elliptic partial differential equation (PDE) for the unknown function u. The problem of prescribing Gaussian curvature thus amounts to studying the existence, uniqueness or multiplicity, and classification of solutions u of (1.2) for the given K. A particularly interesting aspect of the classification problem is the question under which conditions radial symmetry of the prescribed Gauss curvature function K implies radial symmetry of the classical surface $S_g = (\mathbb{R}^2, g)$ and under which conditions radial symmetry is broken. Notice that the inverse problem may not have a solution. In particular, when considered on $\mathbb{S}^2$ instead of $\mathbb{R}^2$, there are so many obstructions to finding a solution u to (the analog of) (1.2) for the prescribed K that Nirenberg was prompted many years ago to raise the question, Which real-valued functions K are Gauss curvatures of some surface S_g over $\mathbb{S}^2$? For Nirenberg's problem, see [4], [6], [9], [10], [11], [33], [36], [38], [44], [45], [47], [48], [50], and [51]. For related works on other compact 2-manifolds, see, for example, [26] and [60]. In this work, we are interested in the prescribed

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