PAIR CORRELATION OF VALUES OF RATIONAL FUNCTIONS (mod p)

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1. Introduction. The problem of understanding the distribution of fractional parts of polynomials has a long history. Weyl [19] proved that, for any polynomial $f(x) = \alpha_d x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_1 x$ having at least one irrational coefficient, the sequence of fractional parts $(\{f(n)\})_{n\in\mathbf{N}}$ is equidistributed in the unit interval $[0,1)$. Since then, many other results on the distribution of fractional parts of polynomials, particularly on "small" fractional parts, have been obtained (see, e.g., Schmidt [15] and Baker [1]).

A different aspect of the random behavior of such a sequence has attracted attention recently, namely, the distribution of the spacings between members of the sequence. This problem came up in the context of the distribution of spacings between energy levels of integrable systems (see Berry and Tabor [2]).

If $f(x) = \alpha x$, the spacings are essentially those of the energy levels of a two-dimensional harmonic oscillator (see Pandey, Bohigas, and Giannoni [10] and Bleher [3], [4]). In this case the sequence is not random; in fact, for any α and N the consecutive spacings of $(\{\alpha n\})_{n\in\mathbf{N}}$ take at most three values (see Sós [16] and Swierczkovski [17]).

In the more interesting case $f(x) = \alpha x^d$, $d \geq 2$, the pair correlation problem was investigated by Rudnick and Sarnak [12], who proved that for almost all α the pair correlation function exists and is "Poissonian." The pair correlation function $R_2(x)$ is defined as follows: say, $\mathbf{x} = (x_n)_{n\in\mathbf{N}}$ is a sequence of real numbers equidistributed in the unit interval $[0,1)$. We take the first N members of the sequence, which provide us with a partition of $[0,1)$ in small intervals of length $1/N$ *in average*. Then we count the number of *normalized* differences between members of the sequence which fall (mod $\mathbf{Z}$) in a given interval $I = [a,b]$. Thus, we consider the quantity

$$R_2(N,\mathbf{x},I) := \frac{1}{N}\#\big\{1 \leq i \neq j \leq N;\ N(x_i - x_j) \in I + N\mathbf{Z}\big\}. \tag{1.1}$$

Note that for $I = [-s,s]$ one gets

$$R_2\big(N,\mathbf{x},[-s,s]\big) = \frac{1}{N}\#\left\{1 \leq i \neq j \leq N;\ \|x_i - x_j\| \leq \frac{s}{N}\right\}.$$

This is the formula that defined the pair correlation in [12].

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