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ON A REFINEMENT OF WARING'S PROBLEM

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§1. Introduction

§1.1. The problem and the result. In this paper $\mathbb{N}_0$ denotes the set of nonnegative integers. A subset S of $\mathbb{N}_0$ is a basis of order r if every positive integer can be represented as the sum of r elements in S. The most trivial basis is $\mathbb{N}_0$ itself, while the most interesting ones are probably the sets of kth powers ($k = 2, 3, \dots$). Waring's classical problem (first solved by Hilbert [Hil]) asserts that for any fixed k and s sufficiently large, every positive integer can be represented as a sum of s kth powers. For instance, every positive integer is a sum of four squares, nine cubes, and so on. Using Hardy-Littlewood's circle method, one can actually estimate the number of representations. The following theorem is classical (see [Vau] and [Nat2], for instance).

THEOREM 1.1. *For any fixed $k \geq 2$, there is a constant $s_1(k)$ such that if $s > s_1(k)$, then $R^s_{\mathbb{N}_0^k}(n)$, the number of representations of n as a sum of s kth powers, satisfies*

$$R^s_{\mathbb{N}_0^k}(n) = \Theta\big(n^{s/k-1}\big)$$

for every positive integer n.

Theorem 1.1 (proved by Vinogradov and also many others) shows that the set $\mathbb{N}_0^k$ of kth powers is not only a basis but also a very rich one; that is, the number of representations of n is huge for all n. (Theorem 1.1 also holds for $k = 1$ as a trivial fact.) A natural question is whether $\mathbb{N}_0^k$ contains a subset X that is a *thin* basis (sometimes we call X a subbasis of $\mathbb{N}_0^k$); that is, for every positive integer n, $R^s_X(n)$ is positive but *small*. The study of thin bases was started by Rohrbach and Sidon in the 1930s and has since then attracted considerable attention from both combinatorialists and number theorists (see [Erd], [EN], [CEN], [Ruz], [Nat1], [Zöl1], [Zöl2], [Wir], [Spe], [ER], [ET], and [HR]).

How small? one may wonder. A very old, but still unsolved, conjecture of Erdős and Turán [ET] states that if X is a basis of order 2, then $\limsup_{n\to\infty} R^2_X(n) = \infty$. Since this conjecture is commonly believed to be true even for arbitrary order, the best we can hope for is to prove that there exists $X \subset \mathbb{N}_0^k$ such that $R^s_X(n)$ is a positive but slowly increasing function in n. The objective of this paper is to prove the following theorem.

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