TOPOLOGICAL DEGREE FOR MEAN FIELD EQUATIONS ON S^2

CHANG-SHOU LIN

1. Introduction. Let (S^2, g_0) be the unit sphere of $\mathbf{R}^3$ equipped with the metric g_0 induced from the flat metric of $\mathbf{R}^3$. For a positive smooth function f on S^2, we consider the nonlinear equation

$$\Delta\phi+\rho\left(\frac{f(y)e^{\phi}}{\int_{S^2} f(y)e^{\phi}d\mu}-\frac{1}{4\pi}\right)=0 \quad \text{on } S^2, \tag{1.1$_\rho$}$$

where Δ is the Beltrami-Laplace operator of $(S^2, g_0), d\mu$ is the volume form with respect to g_0, and $\rho > 0$ is a constant. Obviously, equation $(1.1)_\rho$ is invariant under adding a constant c. Hence, we always seek solutions of $(1.1)_\rho$, which are normalized by

$$\int_{S^2} \phi(y)\, d\mu(y) = 0. \tag{1.2}$$

Equation $(1.1)_\rho$ is called the mean field equation because it often arises in the context of statistical mechanics of point vortices in the mean field limits. Recently, there has been interest in $(1.1)_\rho$ because it also arises from the Chern-Simons-Higgs model vortex theory when some parameter tends to zero. (For these recent developments, we refer the readers to [5], [2], [3], [10], [11], [13], [14], [18], [19], [21], [22], and the references therein.)

Clearly, equation $(1.1)_\rho$ is the Euler-Lagrange equation of the nonlinear functional

$$J_\rho(\phi) = \frac{1}{2}\int_{S^2} |\nabla\phi|^2\, d\mu - \rho \log\left(\int_{S^2} f(y)e^{\phi}\, d\mu\right) \tag{1.3}$$

for $\phi \in H^1(S^2)$ satisfying (1.2). Here $H^1(S^2)$ denotes the Sobolev space of functions with L^2-integrable first derivatives. For $\rho < 8\pi$, $J_\rho(\phi)$ is bounded below, and the infinimum of J_ρ can be achieved by the well-known Moser-Trudinger inequality. However, for the case $\rho \geq 8\pi$, the existence of solutions to $(1.1)_\rho$ is much more delicate. Recently, under some conditions on f, the existence of an infinimum of $J_{8\pi}$ has been proved by [10] and [18]. However, the existence of solutions to $(1.1)_\rho$

Received 3 August 1999. Revision received 28 January 2000.
2000 *Mathematics Subject Classification*. Primary 35J60.