Vol. 104, No. 3 DUKE MATHEMATICAL JOURNAL © 2000

TRACES OF INTERTWINERS FOR QUANTUM GROUPS AND DIFFERENCE EQUATIONS, I

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0. Introduction. This paper begins a series of papers whose goal is to establish a representation-theoretic interpretation of the quantum Knizhnik-Zamolodchikov-Bernard (qKZB) equations and to use this interpretation to study solutions of these equations. It was motivated by the recent work on the qKZB equations in [Fe], [FeTV1], [FeTV2], [MuV], [FeV2]–[FeV5] and by the theory of "quantum conformal blocks" that began with the classical paper [FR].

0.1. The qKZB equations in [Fe] are difference equations with respect to an unknown function $f(z_1,\dots,z_N,\lambda,\tau,\mu,p)$ with values in $V_1\otimes\cdots\otimes V_N\otimes V_N^*\otimes\cdots\otimes V_1^*$, where V_i are suitable finite-dimensional representations of the quantum group $U_q(\mathfrak{g})$ ($\mathfrak{g}$ is a simple Lie algebra), $z_i, p, \tau\in\mathbb{C}$, and λ, μ are weights for $\mathfrak{g}$.

The qKZB equations are a q-deformation of the Knizhnik-Zamolodchikov-Bernard (KZB) differential equations and an elliptic analogue of the quantum Knizhnik-Zamolodchikov (qKZ) difference equations, which are, in turn, generalizations of the usual (trigonometric) Knizhnik-Zamolodchikov (KZ) equations.

It is proved in [FeTV2] (using an integral representation of solutions) that for $\mathfrak{g}=\mathrm{sl}_2$ the monodromy of the qKZB equations is given by the dual qKZB equations, which are obtained from the qKZB equations by interchanging (λ,τ) with (μ,p). This fact generalizes the monodromy theorems for the KZB and qKZ equations: the monodromy of the KZB differential equations is the trigonometric degeneration of the qKZB equations (which involves dynamical R-matrices without spectral parameter) (see, e.g., [K3]), and the monodromy of the qKZ equations is given by elliptic dynamical R-matrices (but there is no difference equation) (see [TV1], [TV2]; see also [FR]).

The self-duality of the qKZB equations leads one to expect that they should have symmetric solutions $u_{V_1,\dots,V_N}(z,\lambda,\tau,\mu,p)$, that is, such that $u_{V_1,\dots,V_N}(z,\lambda,\tau,\mu,p) = u^*_{V_N^*,\dots,V_1^*}(z,\mu,p,\lambda,\tau)$, where u^* is the dual of u (considered as an endomorphism of $V_1\otimes\cdots\otimes V_N$). Such a solution u (for $\mathfrak{g}=\mathrm{sl}_2$) was constructed in [FeV3] and [FeTV2] by an explicit integral formula. It is called the universal hypergeometric function. This function has many interesting properties, in particular the $\mathrm{SL}(3,\mathbb{Z})$-symmetry (see [FeV2], [FeV4], and [FeV5]), where the group $\mathrm{SL}(3,\mathbb{Z})$ acts on the lattice $\mathbb{Z}^3$ generated by the periods $1,\tau,p$. A consequence of this symmetry

Received 31 August 1999. Revision received 29 November 1999.
2000 *Mathematics Subject Classification.* Primary 20G42; Secondary 33D52, 37K10.
Etingof's work partially supported by National Science Foundation grant number DMS-9700477.
Varchenko's work partially supported by National Science Foundation grant number DMS-9801582.