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ON THE NUMBER OF NONREAL ZEROS OF REAL ENTIRE FUNCTIONS AND THE FOURIER-PÓLYA CONJECTURE

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This paper is concerned with a general theorem on the number of nonreal zeros of transcendental functions. J. Fourier formulated the theorem in his work *Analyse des équations déterminées* in 1831, but he did not give a proof. Roughly speaking, the theorem states that if a real entire function $f(x)$ can be expressed as a product of linear factors, then we can *count* the nonreal zeros of $f(x)$ by observing the behavior of the derivatives of $f(x)$ on the real axis alone. As we shall see in the sequel, this theorem completely justifies his former argument, by which he tried to prove that the function $J_0(2\sqrt{x})$ has only real zeros. It seems that no complete proof of the theorem is known, and no general theorem has been published that justifies the argument. Later, in 1930, G. Pólya published a paper entitled *Some problems connected with Fourier's work on transcendental equations* [P3]. In this paper, Pólya conjectured two hypothetical theorems that are closely related to Fourier's unproved theorem. In fact, he conjectured three, but he proved that two of them are equivalent to each other. The first hypothetical theorem is a modernized formulation of the theorem, and it justifies Fourier's argument completely. The second conjecture was proved in 1990, but it is impossible to justify the argument using the conjecture alone. In the present paper, we prove Pólya's formulation of the theorem (his first conjecture) as well as its extensions, give a very simple and direct proof of the second conjecture mentioned above, and exhibit some applications of the results. In particular, we completely justify Fourier's argument by our general theorems.

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1. Historical introduction. In this section, we briefly explain our results as well as their background. A *real entire function* is an entire function that assumes only real values on the real axis. Fourier's unproven theorem asserts that we can know the number of nonreal zeros of such real entire functions by counting their *critical points*, which are defined as follows: Let $f(x)$ be a real analytic function defined in an open

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