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ANALOGS OF WIENER'S ERGODIC THEOREMS FOR SEMISIMPLE LIE GROUPS, II

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§0. Introduction. Given a measure-preserving action $T_v : X \to X,\ v \in \mathbb{R}^d$ of the group $G = \mathbb{R}^d$ on a probability space (X, m), and a function $f \in L^1(X)$, consider the averaging operators

$$\pi(\beta_t) f(x) = \frac{1}{\operatorname{vol} B_t} \int_{v \in B_t} f(T_v x)\, dv,$$

where $B_t = \{v \in \mathbb{R}^d, \|v\| \le t\}$.

Wiener's pointwise ergodic theorem asserts that $\pi(\beta_t) f(x)$ converges to a limit as $t \to \infty$ for almost every $x \in X$. The limit is given by the average of f on X, namely, $\int_X f\, dm$, provided the action is ergodic. The main tool used in the proof of this result is Wiener's maximal inequality, which asserts that the maximal function $f_\beta^*(x) = \sup_{t>0} |\pi(\beta_t) f(x)|$ satisfies $m\{x : f_\beta^*(x) \ge \delta\} \le (C/\delta)\|f\|_{L^1(X)}$.

Consider the following generalization of the foregoing setup. Let G be a connected Lie group G, and let K be a compact subgroup. Assume there exists a G-invariant Riemannian metric on the homogeneous space $S = G/K$. The (bi-K-invariant) ball averages β_t on G are defined to be the probability measures

$$\beta_t = \frac{1}{m_G(B_t)} \int_{g \in B_t} \delta_g\, dm_G(g),$$

where m_G is a left-invariant Haar measure on G, $B_t = \{g \in G \mid d(gK, K) \le t\}$, d is the Riemmanian distance on $S = G/K$, and δ_g is the delta measure at g. β_t give rise to canonical averaging operators, denoted $\pi(\beta_t)$, in every measure-preserving action of G. We can now formulate the following problem.

BALL AVERAGING PROBLEM. *Determine whether, for any ergodic measure-preserving action of G on a probability space (X, m), the averaging operators $\pi(\beta_t) f(x)$ converge to $\int_X f\, dm$, for $f \in L^1(X)$, or at least for $f \in L^p(X)$, $p > 1$. Also, determine whether the maximal inequality $\|f_\beta^*\|_{L^p(X)} \le C_p \|f\|_{L^p(X)}$ holds.*

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