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ORTHONORMAL BASES OF EXPONENTIALS FOR THE n-CUBE

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1. Introduction. A compact set Ω in $\mathbb{R}^n$ of positive Lebesgue measure is a spectral set if there is some set of exponentials

$$\mathcal{B}_\Lambda := \left\{ e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda \right\}, \tag{1.1}$$

which when restricted to Ω gives an orthogonal basis for $L^2(\Omega)$, with respect to the inner product

$$\langle f, g\rangle_\Omega := \int_\Omega \overline{f(x)} g(x)\, dx. \tag{1.2}$$

Any set Λ that gives such an orthogonal basis is called a spectrum for Ω. Only very special sets Ω in $\mathbb{R}^n$ are spectral sets. However, when a spectrum exists, it can be viewed as a generalization of Fourier series, because for the n-cube $\Omega = [0,1]^n$ the spectrum $\Lambda = \mathbb{Z}^n$ gives the standard Fourier basis of $L^2([0,1]^n)$.

The main object of this paper is to relate the spectra of sets Ω to tilings in Fourier space. We develop such a relation for a large class of sets and apply it to geometrically characterize all spectra for the n-cube $\Omega = [0,1]^n$.

THEOREM 1.1. *The following conditions on a set Λ in $\mathbb{R}^n$ are equivalent.*

(i) *The set $\mathcal{B}_\Lambda = \{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ when restricted to $[0,1]^n$ is an orthonormal basis of $L^2([0,1]^n)$.*

(ii) *The collection of sets $\{\lambda + [0,1]^n : \lambda \in \Lambda\}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.*

This result was conjectured by Jorgensen and Pedersen [6], who proved it in dimensions $n \le 3$. We note that in high dimensions there are many "exotic" cube tilings. There are aperiodic cube tilings in all dimensions $n \ge 3$, while in dimensions $n \ge 10$ there are cube tilings in which no two cubes share a common $(n-1)$-face; see Lagarias and Shor [9].

In Theorem 1.1, the n-cube $[0,1]^n$ appears in both conditions (i) and (ii), but in functorially different contexts. The n-cube in (i) lies in the space domain $\mathbb{R}^n$ while the n-cube in (ii) lies in the Fourier domain $(\mathbb{R}^n)^*$, so they transform differently under linear change of variables. Thus Theorem 1.1 is equivalent to the following result.

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