VARIATIONAL PROPERTIES OF A NONLINEAR ELLIPTIC EQUATION AND RIGIDITY

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1. Introduction and main results. In this paper, we discuss variational properties of classical solutions of the nonlinear equation

$$\Delta u = -V_u'(u, x_1, \dots, x_n), \tag{1.1}$$

which is the Euler-Lagrange equation of the functional

$$I(u) = \int \frac{1}{2}(\nabla u)^2 - V(u, x_1, \dots, x_n)\, dx_1 \cdots dx_n. \tag{1.2}$$

The main question addressed here is the following: Under what conditions on the potential V are all classical solutions of (1.1) globally minimising for the functional (1.2)? By a global minimiser we mean a smooth function on a domain of $\mathbb{R}^n$ minimising the integral (1.2) over all bounded subdomains with smooth boundaries with respect to smooth functions with the same boundary values.

The motivation for this question comes from variational problems of classical mechanics. In geodesic problems it is well known that all geodesics are globally minimising on manifolds of negative sectional curvature. However, it was shown first by Hopf [12] and Green and Gulliver [9] that the situation is completely different for Riemannian (two) tori or for Riemannian planes that are flat outside a compact set. They proved that for these manifolds, there always exist geodesics with conjugate points, which are therefore nonminimal, unless the metric is flat. We refer the reader to [4] and [6] for higher-dimensional generalisations of Hopf and Green-Gulliver theorems and to [8], [7], [13], and [10] for very important previous developments.

It was observed first in [2] that the Hopf phenomenon is not entirely Riemannian. In [3] it was shown that a similar type of rigidity holds for Newton's equations with periodic or compactly supported potentials.

For the equation (1.1) with periodic potentials, it was shown in [15] and [16] that far-reaching generalisations of Aubry-Mather and KAM (Kolmogorov-Arnold-Moser) theories apply. Using these theories, one can construct families of minimal solutions that form laminations or sometimes even foliations of the configuration torus. It is an interesting open question, however, if it happens that, for all *slope*

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