Comment

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There is a great deal to be admired in the extensive work on chaos that has appeared in recent years, including some startling but simple theorems, and also the best art work produced by mathematics. However, in my opinion, it is often surrounded by an unnecessary amount of hype, considerable zeal and possibly some illogical arguments and confusion.

To simplify this discussion, I will consider only series that are "white chaos" and compare them with iid series. A process will be called "white" if its (estimated) autocorrelations are all zero and thus the (estimated) spectrum is flat, with estimates based on a long realization of the process. White chaos is a deterministic process with these white properties. As an example, I will consider the process generated by

$$x_{t+1} = 4x_t(1 - x_t) \tag{1}$$

with starting value $x_0 = s$, s being the "seed" value. I will also assume that truly stochastic processes exist—an assumption that I think most scientists will accept with probability one. Thus, an iid series y_t exists, and such a series is also obviously white.

Let G_1, G_2 be a pair of generating mechanisms, producing series x_{1t}, x_{2t}; then, it is obviously possible that the two series will have some properties in common, such as zero means and identical (estimated) spectra. Many generating mechanisms can produce series having the white properties, as pointed out in Granger (1983). An example is the bilinear process generated by

$$y_t = \alpha y_{t-1}\varepsilon_{t-2} + \varepsilon_t, \tag{2}$$

where ε_t is zero mean iid. It is clearly possible for a (deterministic) white chaos to have many properties of an iid process. Statisticians are familiar with pseudo-random numbers (prn) generated on computers by a somewhat complex deterministic model. These numbers are chaos of "high dimension," as defined in the papers being discussed, or "space-filling." It is generally agreed that it would take an enormous amount of data—a sample size of billions—to distinguish prn from a true iid. The only questions then worth considering is how to distinguish between a low-dimensional white chaos and an iid series, and thus whether or not white chaos occurs in reality rather than in computer simulations or physics laboratory experiments.

The papers emphasize the similarities between white chaos and iid series, such as the similar appearance of their plots through time or the values taken by statistics such as autocorrelations. The fact that white chaos can look like iid, which can be restated as an iid series that looks like white chaos, has no implication. If two generating mechanisms produce series, each of which has some properties, P, it does not mean that the mechanisms are identical or similar. There is a danger of falling into the famous logical fallacy that says, "If A then B, observe B therefore A." An example would be, "If chaos (A) then positive Lyapunov exponent (B), if data has a positive Lyapunov exponent then it must be chaos," which is seen occasionally in chaos literature but is, of course, false because some stochastic processes, such as an AR(1) with the coefficient larger than one, also have positive Lyapunov exponents. It follows that this exponent cannot be used as a "popular measure of chaos" (Berliner, Section 3) without the added assumption that the process *is* chaos.

A similar problem arises with the interpretation of ergodicity. Let the proportion of time that a series lies in some region R asymptotically tend to a constant, for every R. This asymptotic proportion could be called the likelihood that the series is (eventually) in R. The fact that chaotic series have such likelihood is interesting but not especially surprising. If the series are also assumed, or known, to be stochastic, then these likelihoods can be called probabilities and interpreted in the usual frequency count manner. There is no philosophical problem in doing this. Without the assumption of stochasticity, the likelihood need not be called a probability and then no unnecessary confusion occurs. The likelihood can be put together for different sets R to derive a marginal "distribution" for the series. However, of much greater interest is the joint distribution of a set of adjacent values of the series, which can be derived in a similar manner. Consider a pair of random variables X, Y, with a joint distribution $f(x, y)$. They can be called "singular" if there exists some combination $X - g(Y)$,

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