structure that may be obscured by even small amounts of noise). Smith (1991) has discussed the problem of dimension estimation for systems with this type of observational noise.

A different type of noise results when the system under evolution undergoes perturbations due to some external force or change. The perturbations are then propagated through the system. This also describes the situation of rounding error in numerical simulations of chaotic systems; the original rounding error is repeatedly magnified by the "stretching" behavior of the map and the computed numerical trajectory [called a pseudo-orbit by Hammel et al. (1988)] diverges far from the true path. Hammel, Yorke and Grebogi pointed out that often pseudo-orbits are in fact true orbits corresponding to different initial conditions, but even in the ergodic case, this is not necessarily reassuring. The dyadic map $T(x) = 2x \pmod 1$ on the unit interval is ergodic and chaotic with the uniform distribution as invariant measure. However, all orbits of this map quickly iterate to zero on the computer. These are true orbits of the system; unfortunately, they correspond to initial conditions (dyadic rationals) that are attracted to the fixed point at zero and do not exhibit "typical" system behavior. Thus, in numerical simulations, it is not always easy to determine whether observed behavior is "real" or an artifact of the simulation procedure.

Corless (1991) has looked at the related problem of approximating solutions to differential equations by numerical methods (here again the computed solution may not resemble the intended system; see Hockett, 1990 and Corless, Essex and Nerenberg, 1991) and has proposed an "operational" definition of chaos. He suggests that a system should be considered chaotic if all "nearby" solutions are chaotic (regardless of the actual properties of the system itself). The reasoning here is that perturbations will cause any physical system to be pushed into neighboring states and these should be the real objects of study.

Comment: Inference and Prediction in the Presence of Uncertainty and Determinism

John Geweke

1. INTRODUCTION

The discovery of nonlinear determinism and chaos in physical systems, the study of these phenomena by physicists and mathematicians and their consideration by investigators in a wide array of disciplines have been ably surveyed from the perspective of statistics and probability in these two articles. The authors have indicated clearly that the contributions and relevance of statistical science are still unresolved, and some basic questions are open. Because chaotic dynamics generate realizations that can be characterized as purely random, what is role of stochastic modeling? If observed deterministic nonlinear processes always interact with stochastic processes, then are the conventional tools of statistical inference any less adequate here than elsewhere? The resolution of these issues will take time, and these surveys will contribute to this process by having brought the statistically relevant aspects of nonlinear determinism and chaos to a wider audience.

Independent of how these questions are answered, the models discussed bring to the practical level latent questions about the implications of determinism for the fundamental role that randomness seems to play in so much of statistics. Berliner has discussed these matters in the final section of his contribution. I have found deterministic models an enlightening vehicle for taking up these questions on a practical level, and in these brief remarks I will provide a few illustrations. The next section provides an approach to inference and prediction in the nonstochastic world of the models these authors have discussed. The likelihood function is presented for two simple models in Section 3, and the construction of predictive densities (for the past, as well as the future) is illustrated in Section 4.

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