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ADDENDUM TO WEAK AND STRONG UNIFORM CONSISTENCY OF THE KERNEL ESTIMATE OF A DENSITY AND ITS DERIVATIVES

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This work [4] was concerned with the kernel estimate f_n of a uniformly continuous density f on the real line, defined by

$$f_n(x) = \Sigma_{i=1}^n (nh)^{-1}\delta\{h^{-1}(x - X_i)\}$$

where $X_1, X_2, \cdots$ are i.i.d. random variables with density f, the function δ is a kernel function and $h(n)$ is a sequence of window widths. Under mild conditions on δ, the following theorem was proved, using results on the strong embedding of the empirical distribution function and various theorems on Gaussian processes, including those proved in [3].

THEOREM A. *Suppose δ satisfies the following conditions*:

(1)
(a) *δ is uniformly continuous on* $(-\infty, \infty)$;
(b) *δ is of bounded variation on* $(-\infty, \infty)$;
(c) $\int |\delta(x)|\, dx < \infty$ *and* $\int \delta(x)\, dx = 1$;
(d) $\int |x \log |x||^{\frac{1}{2}} |d\delta(x)| < \infty$.

Suppose that f is uniformly continuous and that $h \to 0$ and $(nh)^{-1} \log n \to 0$ as $n \to \infty$. Then

$$\sup |f_n - f| \to 0 \qquad \text{a.s. } as\ n \to \infty.$$

In [3] the additional condition $\delta(x) \to 0$ as $|x| \to \infty$ was stated; this follows from conditions (a) and (b) above. The same techniques of proof yielded exact rates of weak and strong convergence of $\sup |f_n - \mathbb{E} f_n|$ to zero; in the case of weak consistency the exact best value of the implicit constant in the rate of convergence was provided, while in the strong consistency case bounds on the value of the constant were obtained. See Theorem B of [4] for details; these rates are essential for the proof of the theorem underlying the practical method described in [5]. In addition the estimation of density derivatives was discussed.

It is possible to vary the conditions (1) on δ in Theorem A. A theorem of Bertrand-Retali stated in [1] and proved in [2] has the same conditions on f and h and the same conclusions, under the following assumptions on the kernel δ:

(a) the set of discontinuity points of δ has Lebesgue measure zero;

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