SYSTEMS OF LINEAR EQUATIONS WITH COEFFICIENTS SUBJECT TO ERROR

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1. Introduction. Various scientific problems lead to non-homogeneous systems of n linear equations in n unknowns, in which the $n^2 + n$ coefficients (including "absolute" terms) are subject to error. Such errors may be errors of observation, or errors introduced by rounding off decimal expansions. If the system has a non-vanishing determinant, the ordinary rules yield the solution. But the question arises: how may the possible errors in the coefficients affect the solutions? In particular, one would like to know how to exclude the fatal event that some malicious combination of errors might make the determinant zero. One would further like to have limitations on the solution-errors in terms of maximum coefficient-errors. Considering the coefficient-errors as random variables, one may also inquire as to the probability distributions of the solution-errors.

The principal result obtained in this paper is the Taylor's expansion of the error in any unknown, considered as a function of the $n(n + 1)$ errors in the coefficients. An upper bound is obtained for each term of this series, and the sum of these upper bounds (when convergent) is expressed in closed form. Thus are obtained not only approximations to the maximum error, but an actual upper limit. Convergence of the power series is established for sufficiently small coefficient-errors; "sufficient smallness" is specified in terms of a simple criterion, which simultaneously provides a sufficient condition for the non-vanishing of a determinant with elements subject to error.

These results were obtained before I learned that work had already been done on the problem. The earliest seems to be that of F. R. Moulton [2] in 1913; he found the first order approximation (6) for $n = 3$, and discussed the geometrical reasons for sensitivity. Much later I. M. H. Etherington [1], evidently unaware of Moulton's paper, found the expression for the total error of a determinant whose elements may be in error, and applied this to the present problem. He thus found limits for the first and second order errors, in a rather different form from mine. The probabilistic considerations of section 5 were suggested by Etherington's article. L. B. Tuckerman [3] recently discussed the question of estimating computational errors incurred in the course of solution. He considered only errors of first order.

My original procedure was to compute the terms of the Taylor's series as successive differentials of the unknown, from Cramer's formula. This soon becomes laborious, and I found only the first two terms. The linear matrix equation (4) was then kindly suggested to me by R. Oldenburger. Here (4) is solved by iteration, resulting in a simple recursion formula for successive terms of the Taylor's series.