STATISTICAL TESTS BASED ON PERMUTATIONS OF THE OBSERVATIONS

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1. Introduction. One of the problems of statistical inference is to devise exact tests of significance when the form of the underlying probability distribution is unknown. The idea of a general method of dealing with this problem originated with R. A. Fisher [13, 14]. The essential feature of this method is that a certain set of permutations of the observations is considered, having the property that each permutation is equally likely under the hypothesis to be tested. Thus, an exact test on the level of significance α can be constructed by choosing a proportion α of the permutations as critical region. In an interesting paper H. Scheffé [2] has shown that for a general class of problems this is the only possible method of constructing exact tests of significance.

Tests based on permutations of the observations have been proposed and studied by R. A. Fisher, E. J. G. Pitman, B. L. Welch, the present authors, and others. Pitman and Welch derived the first few moments of the statistics used in their test procedures. However, it is desirable to derive at least the limiting distributions of these statistics and make it practicable to carry out tests of significance when the sample is large. Such a large sample distribution was derived for a statistic considered elsewhere [1] by the present authors.

In this paper a general theorem on the limiting distribution of linear forms in the universe of permutations of the observations is derived. As an application of this theorem, the limiting distributions of the rank correlation coefficient and that of several statistics considered by Pitman and Welch, are obtained. In the last section the limiting distribution of Hotelling's generalized T in the universe of permutations of the observations is derived.

2. A theorem on linear forms. Let $H_N = (h_1, h_2 \dots, h_N)$ $(N = 1, 2, \dots, \text{ad inf.})$ be sequences of real numbers and let

$$\mu_r(H_N) = N^{-1} \sum_{\alpha=1}^{N} \left((h_\alpha - N^{-1} \sum_{\beta=1}^{N} h_\beta \right)^r$$

for all integral values of r. We define the following symbols in the customary manner: For any function $f(N)$ and any positive function $\varphi(N)$ let $f(N) = O(\varphi(N))$ mean that $| f(N)/\varphi(N) |$ is bounded from above for all N and let

$$f(N) = \Omega(\varphi(N))$$

mean that

$$f(N) = O(\varphi(N))$$

and that

$$\lim_N \inf | f(N)/\varphi(N) | > 0.$$