NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

A USEFUL CONVERGENCE THEOREM FOR PROBABILITY DISTRIBUTIONS

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In problems of establishing limiting distributions it is often apparent that the probability density $p_n(x)$ of a random variable X_n has a limit $p(x)$; throughout this paper $n = 1, 2, 3, \cdots$, and all limits are taken as $n \to \infty$. If $p(x)$ is the density of a random variable X, what we really care about then is whether the limits apply to probabilities, which involve integrals of the densities: Does $\lim Pr\{X_n \text{ in } S\} = Pr\{X \text{ in } S\}$ for all[1] Borel sets S, or, does

$$\lim \int_S p_n(x)\, dx = \int_S p(x)\, dx \ ? \tag{1}$$

The question is thus one of taking a limit under an integral sign. Perhaps the most widely used justification of such a process is the following theorem of Lebesgue [1, p. 47; 2, p. 29]: If for a sequence $\{f_n(x)\}$ of integrable functions, $\lim f_n(x) = f(x)$ for almost all x in S, then a sufficient condition that

$$\lim \int_S f_n(x) dx = \int_S f(x)\, dx$$

is that there exist an integrable function $g(x)$ which uniformly dominates the sequence $\{f_n(x)\}$, that is, $|f_n(x)| \le g(x)$ for all n and all x in S, and $\int_S g(x)\, dx < \infty$. For example, in the excellent new treatise by Cramér the limiting form of the t-distribution is treated as follows [1, p. 252; other examples on pp. 369, 371]: For n degrees of freedom the t-variable has the density

$$p_n(x) = c_n(1 + x^2/n)^{-\frac{1}{2}(n+1)}, \tag{2}$$

where

$$c_n = (n\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}(n+1))/\Gamma(\tfrac{1}{2}n). \tag{3}$$

It is shown fairly easily that $\lim p_n(x) = p(x)$, the density of $N(0, 1)$, where

[1] In defining the convergence of a sequence of distributions to the distribution of a discontinuous random variable X it is desirable to modify this requirement so that it is demanded only of sets S which are continuity intervals of X [1, p. 83]. We are concerned here however only with the "absolutely continuous case" where X has a probability density $p(x)$.