ON THE POWER FUNCTION OF THE "BEST" t-TEST SOLUTION OF THE BEHRENS-FISHER PROBLEM

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1. Introduction. The Behrens-Fisher problem is concerned with significance tests for the difference of the means of two normal populations when the ratio of the variances of the populations is unknown. Denote one population by $N(a_1, \sigma_1^2)$ and the other by $N(a_2, \sigma_2^2)$, where the notation $N(a, \sigma^2)$ represents a normal population with mean a and variance σ^2. Let m sample values be drawn from $N(a_1, \sigma_1^2)$ and n sample values from $N(a_2, \sigma_2^2)$ where $m \leq n$. Then Scheffé [1] has shown that certain optimum properties are possessed by a t-test solution he proposed for the Behrens-Fisher problem, in which the numerator of t is based on the difference of the means of the samples while the denominator is based on the square root of a function of the sample values which has a χ^2-distribution with $m - 1$ degrees of freedom. The purpose of this note is to compare the power function of this t-test with the power function of the corresponding most powerful test for the case in which the ratio of variances σ_1^2/σ_2^2 is also known (only one-sided and symmetrical tests are considered). This comparison is made by computing the power efficiency (see section 2 for definition) of Scheffé's test.

It is sufficient to limit power efficiency investigations to one-sided tests. As shown in [2], a symmetrical t-test with significance level 2α has the same power efficiency as the corresponding one-sided t-test with significance level α. Equation (2) of section 2 furnishes an explicit formula whereby approximate power efficiencies can be computed for a wide range of values of α, m, n. Table 1 contains values of (2) for $\alpha = .05$, $.01$ and several values of m and n.

For the situation considered here, a power efficiency of $100r\%$ has the quantitative interpretation that the given test based on samples of size m and n has approximately the same power function as the corresponding most powerful test based on samples of size rm and rn. Intuitively the power efficiency of a test measures the percentage of available information per observation which is utilized by that test.

2. Power efficiency derivations. The basic notion of the power efficiency of a significance test is given in [2]. For the present case the problem is to determine the value r such that a most powerful test of the same hypothesis (same significance level) based on rm and rn sample values will have approximately the same power function as the given t-test based on m and n sample values (from $N(a_1, \sigma_1^2)$ and $N(a_2, \sigma_2^2)$ respectively). Here the value of σ_1^2/σ_2^2 is assumed to be known. Then the power efficiency of the given t-test equals $100r\%$.

If the ratio of variances σ_1^2/σ_2^2 is known, the most powerful significance test (one-sided and symmetrical) for the difference of means of the two normal populations is a t-test where the numerator of t is based on the difference of the