ON RATIOS OF CERTAIN ALGEBRAIC FORMS

By Robert V. Hogg

State University of Iowa

1. Introduction. In an investigation of the ratio of the mean square successive difference to the mean square difference in random samples from a normal universe with mean zero, J. D. Williams [4] proved the rather surprising fact that any moment of this ratio is equal to the corresponding moment of the numerator divided by that of the denominator. Later Tjallings Koopmans [2] and John von Neumann [3] showed independently that this ratio and its denominator are stochastically independent. From this, Williams' theorem is an immediate consequence. In this paper, we determine a necessary and sufficient condition for the stochastic independence of a ratio and its denominator. We then use this condition in our study of certain ratios of algebraic forms.

2. Stochastic independence of a ratio and its denominator. We prove the following theorem for the continuous type distribution. Consider two one-dimensional random variables x and y and their probability density function $g(x,y)$. Let $P(y \leqq 0) = 0$. Assume the moment generating function, $M(u,t) = E[\exp(ux + ty)]$, exists for $-T < u,t < T$, $T > 0$. The theorem is as follows.

Theorem 1. *Under the conditions stated, in order that y and $r = x/y$ be stochastically independent, it is necessary and sufficient that*

$$\frac{\partial^k M(0,t)}{\partial u^k} \equiv \frac{\dfrac{\partial^k M(0,0)}{\partial u^k}}{\dfrac{\partial^k M(0,0)}{\partial t^k}} \frac{\partial^k M(0,t)}{\partial t^k},$$

for $k = 0, 1, 2, \cdots$.

Proof of necessity. If $f(r, y)$ is the probability density function of the variables r and y, it is well known that a necessary and sufficient condition for the independence of the random variables r and y is that $f(r,y) \equiv f_1(r)f_2(y)$, where $f_1(r)$ and $f_2(y)$ are the marginal density functions of r and y respectively. Hence, since $x = ry$,

$$M(u,t) \equiv E[\exp(ury + ty)];$$

or

$$M(u,t) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(ury + ty) f_1(r) f_2(y)\, dr\, dy.$$

By hypothesis, the moments of x of order k exist; so

$$\frac{\partial^k M(0,t)}{\partial u^k} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ry)^k \exp(ty) f_1(r) f_2(y)\, dr\, dy.$$