REMARKS ON A MULTIVARIATE TRANSFORMATION[1]

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The object of this note is to point out and discuss a simple transformation[2] of an absolutely continuous k-variate distribution $F(x_1, \cdots, x_k)$ into the uniform distribution on the k-dimensional hypercube. A discussion of related transformations has been given by P. Lévy [1].

Let $X = (X_1, \cdots, X_k)$ be a random vector with distribution function $F(x_1, \cdots, x_k)$. Let $z = (z_1, \cdots, z_k) = Tx = T(x_1, \cdots, x_k)$, where T is the transformation considered. Then T is given by

$$\begin{aligned} z_1 &= P\{X_1 \leqq x_1\} = F_1(x_1), \\ z_2 &= P\{X_2 \leqq x_2 \mid X_1 = x_1\} = F_2(x_2 \mid x_1), \\ &\vdots \\ z_k &= P\{X_k \leqq x_k \mid x_{k-1} = x_{k-1}, \cdots, X_1 = x_1\} = F_k(x_k \mid x_{k-1}, \cdots, x_1). \end{aligned}$$

One can readily show that the random vector $Z = TX$ is uniformly distributed on the k-dimensional hypercube, for

$$\begin{aligned} &P\{Z_i \leqq z_i ; i = 1, \cdots, k\} \\ &\quad = \int_{\{Z \mid Z_i \leqq z_i\}} \cdots \int d_{x_k} F_k(x_k \mid x_{k-1}, \cdots, x_1) \cdots d_{x_1} F_1(x_1) \\ &\quad = \int_0^{z_k} \cdots \int_0^{z_1} dz_1 \cdots dz_k = \prod_{i=1}^{k} z_i, \end{aligned}$$

when $0 \leqq z_i \leqq 1$, $i = 1, \cdots, k$. Hence $Z_1, \cdots, Z_k$ are uniformly and independently distributed on [0, 1].

Let $X_{(i)} = (X_{1(i)}, \cdots, X_{k(i)})$, $i = 1, \cdots, n$, be a random sample of n vectors from a population with distribution function $F(x_1, \cdots, x_k)$ and let $G(x_1, \cdots, x_k)$ be the corresponding sample distribution function. It has been found that the probability distribution of the Kolmogorov-Smirnov statistic,

$$\max_{x_1, \cdots, x_k} | F(x_1, \cdots, x_k) - G(x_1, \cdots, x_k) |,$$

is not the same for all continuous F when $k > 1$ [2]. The same can be said for the multidimensional von Mises statistic

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (F(x_1, \cdots, x_k) - G(x_1, \cdots, x_k))^2 \, d_{x_1 \cdots x_k} F(x_1, \cdots, x_k).$$

However, it would still be of interest to study the Kolmogorov-Smirnov and von Mises statistics for the case of sampling from a population uniformly distributed on the k-dimensional hypercube. One could test whether the $X_{(i)}$, $i = 1, \cdots, n$, are a sample from a population with distribution function $F(x_1, \cdots, x_k)$ by

[1] Work done under ONR contract.

[2] I have recently learned that J. H. Curtiss and I. R. Savage have also considered this transformation.