A CHARACTERIZATION OF THE GAMMA DISTRIBUTION

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1. Introduction. The sum and the difference of two independently and identically distributed normal variates are uncorrelated and therefore also independent. Conversely, we can conclude from the independence of the sum and the difference of two identically and independently distributed random variables that both these variables are normally distributed. In this manner one obtains a characterization of the normal distribution.

We denote in the following by

$$F(x;\alpha,\lambda) = \begin{cases} 0 & x < 0, \\ \dfrac{\alpha^\lambda}{\Gamma(\lambda)} \displaystyle\int_0^x t^{\lambda-1} e^{-\alpha t}\, dt & x > 0, \end{cases} \tag{1.1}$$

the distribution function of the gamma distribution. The corresponding characteristic function is

$$f(t;\alpha,\lambda) = (1 - it/\alpha)^{-\lambda}. \tag{1.2}$$

Here α and $\lambda > 0$ are two parameters. It is seen easily that α is a scale parameter.

Let X and Y be two identically and independently distributed random variables each having the distribution (1.1). It is known [2] that in this case $U = X + Y$ and $V = X/Y$ are also two independent random variables.

In the present paper, we use this fact to derive a characterization of the gamma distribution which is similar to the characterization of the normal distribution mentioned above. Our result can be formulated in the following manner.

Theorem. *Let X and Y be two nondegenerate and positive random variables, and suppose that they are independently distributed. The random variables $U = X + Y$ and $V = X/Y$ are independently distributed if and only if both X and Y have gamma distributions with the same scale parameter.*

In the following it will be convenient to introduce the random variable

$$W = 1/(1 + V) = Y/(X + Y). \tag{1.3}$$

The random variables U and W are both nonnegative. Moreover, W is a bounded random variable and

$$0 \leqq W \leqq 1. \tag{1.4}$$

2. Analytic properties of the characteristic functions of X, Y, U, and W. In this section we consider the characteristic functions of these random variables and investigate in particular whether the integrals defining these functions exist

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