THE MEAN AND VARIANCE OF THE MAXIMUM OF THE ADJUSTED PARTIAL SUMS OF A FINITE NUMBER OF INDEPENDENT NORMAL VARIATES

By M. E. Solari and A. A. Anis

Chelsea Polytechnic, London

1. Introduction. In planning the storage capacity of a reservoir it is desirable to avoid in so far as is practicable both the loss of water that occurs if the reservoir overflows and the harm that is done if the reservoir is empty when water is needed. Hurst [1] on the basis of data from a long series of annual totals of river discharges has discussed the relation between the capacity, the inflow and its variability, and the draft from a reservoir. In the present paper the theoretical analysis of the problem as studied by Anis and Lloyd is carried further.

If, for a period of n years, the annual increment of inflow minus draft is represented by the variable X_i ($i = 1, \cdots, n$) and the partial sums of these increments by $S_r = \sum_{i=1}^{r} X_i$ ($r = 1, \cdots, n$), then the maximum U_n over the n-year period of these S_r is the maximum accumulated storage when there is no deficit, their minimum L_n gives the maximum accumulated deficit when there is no storage, and their range $R_n = U_n - L_n$ gives the capacity necessary to avoid the two difficulties mentioned above. Anis and Lloyd [3] have studied the distribution of U_n and R_n for the idealized case in which the X_i are taken as independent standard normal variables and have shown that, for any $n \geqq 2$, the expected value of the maximum is $(2\pi)^{-1/2} \sum_{s=1}^{n-1} s^{-1/2}$ and hence that the asymptotic value of the mean range, which is twice that of the maximum, agrees with the value $2[(2/\pi)n]^{1/2}$ obtained by Feller [2]. Furthermore Anis [4] has shown the second moment about the origin of the maximum to be

$$\frac{n+1}{2} + \frac{1}{2\pi} \sum_{s=2}^{n-1} \sum_{t=1}^{s-1} t^{-\frac{1}{2}}(s - t)^{-\frac{1}{2}}$$

and has obtained [5] a recurrence relation for computing moments of higher order by means of which he has tabulated the values of the first four moments for $n = 2, 3, \cdots, 15$.

However, from both the engineering and the statistical point of view it is sometimes desirable to separate the effect of inflow and draft, since the latter may be controlled in such a way that the former is the decisive random variable. In his paper Hurst considered the effect that would have been obtained by a rule of release which made the annual draft equal to the mean annual inflow for the n-year period, $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$, so that the accumulation after r years became the adjusted partial sum $S_r' = \sum_{i=1}^{r} X_i - r\bar{X}_n$. For these adjusted partial sums Hurst and Feller both obtained $[(\pi/2)n]^{1/2}$ for the asymp-

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