THE LAGRANGIAN MULTIPLIER TEST

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1. Introduction. One of the problems which occurs most frequently in practical statistics is that of deciding, on the basis of a number of independent observations on a random variable, whether a finite dimensional parameter involved in the distribution function of the random variable belongs to a proper subset ω of the set Ω of possible parameters. Naturally this problem has received considerable attention and the main method which is currently applied in dealing with it is the well-known Neyman-Pearson likelihood ratio test. Direct application of this test involves finding the supremum of the likelihood function in the set ω and this in turn often involves the solution of restricted likelihood equations containing a Lagrangian multiplier. And the same set of of equations has to be solved if, irrespective of the likelihood ratio test, it is desired to obtain a maximum likelihood estimate in the set ω of the unknown parameter. Rather surprisingly, since the problem is of such frequent occurrence, little seems to have appeared in statistical literature on such restricted maximum likelihood estimates, the main results in this field being contained in a recent paper by Aitchison and Silvey [1].

In this paper the authors introduced, on an intuitive basis, a method of testing whether the true parameter does belong to ω, this method being based on the distribution of a random Lagrangian multiplier appearing in the restricted likelihood equations. It is the object of this present paper to discuss this Lagrangian multiplier test. In order to do so, it is necessary to consider how the results of the previous paper must be modified when the true parameter does not belong to the set ω, because only in this way can we obtain any notion of the power of the test. Discussion of this point forms the initial part of the present paper. We will then show the connection between the Lagrangian multiplier test and the likelihood ratio test. Finally, since often in practice situations arise where the information matrix is singular, we will consider how the Lagrangian multiplier test must be adapted to meet this contingency.

The approach adopted by Aitchison and Silvey [1] in the discussion of restricted estimates is essentially Cramér's approach [4] to maximum likelihood estimates, i.e., attention is concentrated on solutions of the likelihood equations rather than on genuine maximum likelihood estimates. Such an approach is really unsuitable in the present instance where we do not necessarily assume that the true parameter does belong to the subset ω. And we will use instead the method used by Wald [7] in his discussion of the consistency of maximum likelihood estimators. As has been pointed out by Kraft and Le Cam [5], Wald's approach to unrestricted maximum likelihood estimation is much more illumi-

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