A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF CONSISTENT ESTIMATES

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1. Introduction. Let $\mathfrak{X}$ be an arbitrary set and let $\mathfrak{A}$ be a σ-field of subsets of $\mathfrak{X}$. Let $\mathfrak{P}$ be the family of all probability measures on $\mathfrak{A}$. Let Θ be a topological space which is homeomorphic to a subset of the cube $K = J^{N_0}$, the product of a countable family of copies of the interval $J = [0, 1]$.

Let $\mathfrak{D}$ be a subset of $\mathfrak{P}$ and let $\varphi : P \rightarrow \varphi(P)$ be a function defined on $\mathfrak{D}$ and taking its values in Θ.

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent identically distributed variables taking their values in $\mathfrak{X}$ and distributed according to some $P \varepsilon \mathfrak{D}$. Our purpose is to give a necessary and sufficient condition for the existence of consistent estimates of the function $\varphi(P)$.

More precisely, the problem can be described as follows. For each integer n let $\mathfrak{X}^n$ be the product of n copies of $\mathfrak{X}$, let $\mathfrak{A}^n$ be the σ-field product of n copies of $\mathfrak{A}$ and let P^n be the measure defined on $\mathfrak{A}^n$ by the product of n copies of P.

Let $\mathfrak{F}$ be an arbitrary family of subsets of $\mathfrak{D}$. If θ and θ' are elements of the cube K let θ_i and θ_1' be their ith coordinates in K and let $\rho(\theta, \theta')$ be the distance

$$\rho(\theta, \theta') = \Sigma \frac{1}{2^i} \mid \theta_i - \theta_i' \mid . \tag{1}$$

By assumption the distance ρ defines on $\Theta \subset K$ its original topology.

Let $\mathfrak{B}$ denote the σ-field of Borel subsets of Θ (or K). We shall say that φ is $\mathfrak{F}$-consistently estimable if there is a sequence $\{T_n\}$ with the following properties:

(1) The function T_n is a measurable map from $\{\mathfrak{X}^n, \mathfrak{A}^n\}$ to $\{\Theta, \mathfrak{B}\}$.

(2) For every $\epsilon > 0$ and $P \varepsilon \mathfrak{D}$ let $V(P\ \epsilon)$ be the sphere set of elements of Θ whose distance to $\varphi(P)$ is not larger than ϵ. Then for every $\epsilon > 0$ and every $F \varepsilon \mathfrak{F}$ the quantity

$$\sup_{P \varepsilon F} P^n[T_n \not\varepsilon V(P, \epsilon)]$$

tends to zero as n tends to infinity.

The explicit purpose of the present paper is to give a characterization of the functions φ which are $\mathfrak{F}$-consistently estimable.

The terminology and results of a topological nature used in this paper can be found in either [1] or [2]. The concept of a precompact uniform structure, neces-

Received March 23, 1959; revised August 3, 1959.

[1] Fellow of the A. P. Sloan Foundation.

[2] This paper was prepared with the partial support of the Office of Ordnance Research, U. S. Army under Contract DA-04-200-ORD-171, Task Order 3.

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